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THE BAKERIAN LECTURE.—"On the Forms of Equipotential Curves and Surfaces and on Lines of Flow." By W. G. ADAMS, M.A., F.R.S., Professor of Natural Philosophy and Astronomy in King's College, London. Received February 18, 1875*.

[PLATES 1 & 2.]

When any portion of a conducting-wire which joins the two poles of a voltaic battery is increased in size, the resistance of the circuit is diminished and a greater current flows through the wire; and as any increase in the cross section of any portion of the wire causes a corresponding increase in the current, it is clear that there is no part of the conducting-wire through which some portion of the current does not flow from the one pole to the other; and the outer boundary of such a wire will be a line of flow.

A line of flow is a line drawn in a conductor which indicates at every point of it the direction of the current at that point. A portion of the conductor completely enclosed by lines of flow may be termed a tube of flow.

If a portion of the wire be beaten out into a flat disk, or if a sheet of metal or tinfoil be employed, it will still be true that there is no part of it to which the lines of flow do not extend, although the amount of current will be weak at points of the disk which are far removed from the shortest line of flow.

If instead of a variable wire or disk the circuit consist in part of a conducting liquid whose cross section is greater than the surface of the electrodes in the liquid, the lines of flow extend throughout the liquid; and an increase in the cross section of the liquid diminishes to some extent the resistance of the circuit.

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* Read February 25, 1875. See vol. xxiii. p. 280.

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The lines of flow diverge from the electrodes or points where the current enters and leaves the liquid, and lie closer and closer together as they lie nearer to the shortest line of flow joining the electrodes.

The whole of the conductor, whether it be a disk or a liquid, may be regarded as being made up of tubes of flow, or of bundles of lines of flow, each tube ending in conical points filling up the same solid angle at the electrodes, and conveying a given quantity of electricity from one battery-electrode to the other.

These tubes increase in diameter at points further and further from the electrodes; and the strength of current at a given point (*i. e.* the quantity of current flowing through a perpendicular section of unit area) will be inversely as the cross section of the tube of flow passing through that point. In passing from one electrode to the other along any line of flow there will be a gradual fall of potential, and the lines of flow will be cut at right angles by surfaces in which the potential is the same for every point of the same surface.

On attaching two wires to a galvanometer, and placing their free ends on the same equipotential surface, no current will flow through the galvanometer.

If, then, two electrodes in connexion with the two poles of a battery be fixed in contact with two points of a disk of tinfoil, or with two points in a conducting liquid, and two other electrodes be attached to a delicate galvanometer, by keeping one of these fixed and tracing with the other we may determine the forms of equipotential curves or surfaces in the disk of tinfoil or in the liquid by joining all the positions of the tracing-pole which give no current through the galvanometer.

It will be convenient to call the electrodes which are connected with the battery the battery-electrodes, and those connected with the galvanometer the galvanometer-electrodes. By means of the tracing-electrode points of the same potential as the fixed galvanometer-electrode may be marked on the tinfoil; and the curve joining them will be an equipotential curve.

Experimental Work.

About five years ago, at the suggestion of Mr. J. T. Bottomley, the tracing out of equipotential curves on sheets of tinfoil was first set as an exercise for our more advanced students in the Physical Laboratory. I have been assisted in the experimental work contained in this paper by Messrs. Jacob, Cochrane, Day, and Harrison, students in the Physical Laboratory of King's College.

To trace the curves, one electrode of a Thomson's reflecting galvanometer was attached to a small screw or pin, fixed in contact with or passing through the tinfoil disk; and the other galvanometer-electrode was attached to a small tube of the same size as the screw, with the end of which contact could be made at any point of the disk. In the centre of this small tube a needle was held by a spring; and when the required point was found, by pressing down the spring a hole could be made in the tinfoil, thus marking the position of the tracing-electrode.

The best form of contact is probably by means of needle-points, on which shoulders of metal 2 or 3 millims. in diameter are soldered, and which are pressed tight on the tinfoil. By placing a sheet of paper underneath the tinfoil disk the forms of equipotential curves are at once pricked out, and may afterwards be drawn. For illustration in lectures the sheet of tinfoil may be placed in front of a lamp, and the forms of equipotential curves or lines of flow may be thrown on a screen. If the curves be traced on a circular disk of the size of or smaller than the condensing-lens, the whole series of equipotential curves on it may be thrown upon the screen at the same time.

Case 1. Plate 1. fig. 1 represents a sheet of tinfoil 310 millims. square, in which A and B, the battery-poles, are 126 millims. apart; and the line AB is nearly parallel to a side, and passes through the centre O of the square; the point O is equidistant from the two poles.

Not far from the centre of the sheet, and in the smaller curves through an angle varying from 60° to 90° about the electrodes, these curves coincide with circles; and in other parts of the curves, when the influence of the edge is taken into account, by supposing the distribution due to charges in the positions of the electric images, as will be afterwards explained, the agreement with the curves as given by theory is remarkably exact.

If a large sheet of tinfoil be taken, and the battery-electrodes be placed far away from the edge of the sheet, then at all points not near the edge of the sheet the forms of the equipotential curves will be very nearly the same as in a sheet of infinite extent.

In all such cases (as will be afterwards shown) the equipotential curves, when there are only two battery-electrodes in connexion with the sheet, are circles having their centres on the straight line passing through the two electrodes; and the lines of flow are also arcs of circles which pass through the two poles.

Case 2. Plate 2. fig. 2 represents a circular sheet of tinfoil, 210 millims. in diameter, with the electrodes on the circumference, and at a distance from one another equal to the radius. The electrodes were small binding-screws placed as closely as possible to the edge of the disk. The differences of potential between two successive equipotential curves have been measured by the deflections of the needle of the galvanometer.

The deflections were as follows:—

From *k* to *a*	150
„ *a* to *b*	50
„ *b* to *c*	50
„ *c* to *e*	50
„ *e* to *f*	50

From f to g	50
„ g to h	80

It will be seen that the fall of potential from k to d is greater than the fall of potential from d to h. This may arise from a difference in the resistances of the contacts with the two battery-electrodes.

The radii of the circles are 28, 56, 290, 82, 28, and 12 millims., beginning from the point k; and the distances between them, measured along the line joining the electrodes, are 20, 13, 15, 19, 15, 10, and 12 millims.

The distances 13, 15, 19, 15, 10 correspond to equal differences of potential; and hence the resistances of the portions of the disk between these consecutive equipotential curves are equal to one another. In this case there was considerable resistance between the binding-screw and the tinfoil disk at the point of contact; but this does not alter the forms of the equipotential curves.

Case 3. Plate 2. fig. 3 represents a large sheet of tinfoil 18 inches square, with one electrode in the centre, by which the current enters the sheet, and four similar electrodes at four corners of a square, each being 3 inches from the central electrode, by which the currents leave the sheet. The electrodes were needles, with shoulders of brass 3 millims. in diameter soldered on them. The four negative electrodes may be united together beneath the board on which the tinfoil is placed, by strips of copper screwed to the electrodes by a small nut on each needle. On the needles which pass through the tinfoil are shoulders which come down tight on the tinfoil so as to make good contact. For these curves two cells of Grove were used; and the difference of potential between two successive curves causes a deflection of 50 divisions of the scale. The resistances of the portions of this disk between successive equipotential curves are equal to one another.

Case 4. The curves in Plate 2. fig. 4, lying within the octant BAH, are equipotential curves, when one positive electrode A is at the corner of a square sheet of tinfoil of which AH and AM are the edges, and one negative electrode at B, at a distance of 3 inches from A, the line AB bisecting the angle between the two edges. The curves between the lines AB and AM have not been drawn in the figure.

The potential at any point in this sheet is that due to the electrodes, together with the three images of each formed by the rectangular edges; and hence the curves should be the same as in the previous case. On comparing the curves it will be seen that that is the case.

The curves, with two exceptions, are drawn at distances corresponding to equal differences of potential; so that, omitting the interpolated curves, the resistances of the portions of the sheet between two consecutive equipotential curves are equal to one another.

This figure also represents the equipotential curves for a square sheet BAB_1, of which AB and AB_1 are the edges, with one positive electrode at A, and two negative electrodes at B and B_1 on the edges of the sheet.

We may also regard the case with one positive electrode at the centre and four negative electrodes at the corners of a square as equivalent to two sets, each set consisting of one positive and two negative electrodes, one on each side of it, at equal distances along the same straight line on a sheet either unlimited or limited by that straight line.

Case 5. The curves for this arrangement of electrodes are drawn in Plate 1. fig. 5; the distance from the positive to each negative electrode is 76 millims., or 3 inches, as in Cases 3 and 4, the electrodes being near the centre of a very large sheet of tinfoil.

Taking the curve which cuts the axis at a distance of 54 millims. from the centre, and at distances of 1 millim. on either side, the distances r_1, r_1' from the negative electrodes to the several points on the curve differ by the quantities in the following Table.

Values of $(r_1 - r_1')$.

For 53 millims.	For 53·75 millims.	For 54 millims.	For 55 millims.
106	108	109	110
105	109	109	110
104	108	110	110
104	108	109	110½
104	108	112	110
101	108	114	111
99			
97			

The curve drawn between those at 53 and 54 millimetres was drawn as nearly as possible at a distance of 53¾ millims. from the centre.

The result of this case shows that in the case of one and four electrodes (Plate 2. fig. 3) we may expect the curves which cut the axis at a distance of about 54 millims. from the centre to be hyperbolas. The fifth curve from the centre is in the position of the rectangular hyperbola, having its foci in the positions of the negative electrodes; and we find, on measuring this curve as well as the curve on the outside of it, that near the vertex the curves are accurately hyperbolas. This is also true of the corresponding curves in fig. 4. The curves first drawn in fig. 5 were drawn at equal distances of 10 millims. apart along the axis, reckoning from the centre; and the differences of potential for these curves, reckoned from the centre, are proportional to the numbers

138, 85, 88, 100, 80,

138 including the effect due to contact of the electrode. Other curves were afterwards interpolated in the neighbourhood of the position of the rectangular hyperbola and around the negative electrodes.

The sheet of tinfoil in the last three cases was sufficiently large for the curves in the neighbourhood of the axis to be similar to those which could have been traced on a sheet of infinite extent.

The third equipotential curve from the negative pole C is the rectangular hyperbola, and its vertex O divides the distance AC; so that

$$AO \text{ is to } AB \text{ as } 1 \text{ to } \sqrt{2}.$$

AO is equal to 53·75 millimetres.

Case 6. Plate 2. fig. 6 represents the case of a circular disk where the current enters at the edge and leaves at the centre of the disk.

Around the centre the curves are very nearly ellipses of small eccentricity, the focus being at the centre of the disk. If a is the radius of the disk and r the distance from the centre, the eccentricity is $\frac{2r}{a}$.

The curve cutting the axis at the point L at a distance of 16 millims., *i. e.* $(3-2\sqrt{2})\,a$ from the centre, has two branches cutting one another at right angles at the point Q, and each cutting the edge of the disk at an angle of 45°, the radius of the disk being 3·75 inches.

When the fixed galvanometer-electrode is at L, it is difficult to find a succession of points forming a continuous equipotential curve; the tracing-electrode may at one time be placed on the boundary of the shaded portion of the figure, and at another may be placed on the axis near the point Q, without causing any current through the galvanometer.

On placing the fixed galvanometer-electrode at Q, the tracing-electrode marks out two straight lines in the neighbourhood of that point of the same potential as the point Q, and each cutting the edge of the disk at an angle of 45° at that point.

The uncertainty in tracing this equipotential curve is explained, as will be presently shown (p. 25), by the fact that each of the galvanometer-electrodes was rather more than 1 millimetre in diameter.

The equipotential curves which lie further from the centre cut the edge of the disk at right angles.

To determine experimentally the Lines of Flow and the Equipotential Surfaces in Space of three dimensions.

If two platinum wires sealed in glass tubes, with only a short piece of wire projecting from the sealed end, be immersed in a liquid, the other ends being connected with the poles of a battery, we shall have a close approximation to the case of currents flowing from one point to another within a liquid; and by means of two other platinum wires similarly arranged but attached to a galvanometer, we may trace out the forms of equipotential surfaces within the liquid.

If dilute sulphuric acid be employed there will be polarization on the electrodes; but by reversing the current alternately, and making contact only for a short time, the polarization may be kept small on the galvanometer-electrodes, provided they are not moved far away from the same equipotential surface.

After a few preliminary experiments to determine how far the method was practicable, I began a definite series of experiments in March 1872. For the experiments in dilute sulphuric acid, in sulphate of copper, and in sulphate of zinc, I have employed a rectangular wooden box, 1 foot long, 8 inches broad, and 8 inches deep. On the edges of the box are fastened paper millimetre-scales; a piece is cut out of the middle of the ends of the box, and a sliding piece fitted in to carry the battery-electrodes. These sliding pieces are capable of motion parallel to the sides of the box, so as to place the battery-electrodes at different distances from one another (see Plate 1. fig. 8). The galvanometer-electrodes were placed firmly in brass tubes, which were accurately placed on **T** pieces of wood so as to be in a line with the point of intersection of two edges of the **T** piece, and to be vertical when the **T** pieces are placed on the edges of the box. By this means the rectangular coordinates of the point could at once be read off on the sides and ends of the box.

The first experiments were made with the points at a depth of 10 centims. below the surface, the box being nearly full of liquid.

In making the experiments the current was reversed, and the readings of the galvanometer taken on both sides of zero for each position of the electrodes.

The battery employed was 20 Leclanche cells, the resistance of each cell being nearly 3 ohms, and the electromotive force about $1\frac{1}{2}$ of Daniell's cell. The strength of the battery-current was measured by a tangent-galvanometer, of the form of Helmholtz's double galvanometer; the deflection of the needles during these first experiments was generally 46°. The galvanometer-electrodes could be brought up to within 1 centim. of the centre of the box.

The object of the first experiments was to determine the changes of potential for equal changes of distance in a direction parallel to the sides of the box.

The battery-electrodes were placed at 2 millims. distance from the end of the box, *i. e.* at a distance of 300 millims. from one another. The line joining them was one axis of coordinates, the other axis being the upper edge of one end of the box. One of the galvanometer-electrodes was placed at the point (150, 50), and the other was first placed at (30, 50), and moved 1 centim. at a time in a direction parallel to the sides of the box. The deflections were taken at each point, and are recorded in the following Table:—

Coordinates of tracing-electrode.	Mean deflection.	Diff.
50, 30	76	
50, 40	66	10
50, 50	61	5
50, 60	55	6
50, 70	50	5
50, 80	44	6
50, 90	36	8
50, 100	30	6
50, 110	24	6
50, 120	18	6
50, 130	13·5	4·5
50, 140	7	6·5
50, 150	2·5	4·5

Coordinates of tracing-electrode.	Mean deflection.	Diff.
50, 150	2·5	
50, 160	− 6	8·5
50, 170	−11·5	5·5
50, 180	−16	4·5
50, 190	−22	6
50, 200	−28	6
50, 210	−34	6
50, 220	−40	6
50, 230	−47	7
50, 240	−52	5
50, 250	−60	8
50, 260	−68	8
50, 270	−74	6
50, 30	74	
50, 50	60	
50, 150	2	

The column of differences gives the differences of potential between the equipotential surfaces passing through the successive positions of the tracing-electrode. These positions are points at equal distances of 10 millims. apart, along a horizontal straight line parallel to the sides of the box at a distance of 50 millims. from the axis.

The difference between the mean and each of the two deflections to right and left may be taken to be the current due to polarization. This was very small, the deflections being 6 at the beginning, 6·5 at the middle, and 10 at the end of the experiments.

Toward the end of the experiments, at (250, 50) there was an increase of 2 in the deflections because the current was kept on for five minutes. From these results it appears that with momentary currents and reversing at each observation, the variation in the polarization current, even in dilute sulphuric acid, may be kept very small. The sudden change in the polarization at (50, 250) reduced the deflection of the tangent-galvanometer from 46° to 41°.

The value of the deflection recorded is the mean of two or more deflections in each direction, but these were very generally the same.

To reduce, if possible, the effect of polarization, I employed sulphate of copper and gutta-percha-covered copper wire 3·6 millims in diameter, with 4 millims. projecting beyond the covering for the battery-electrodes. In the galvanometer-electrodes, which were of the same covered wire, there were 2 millims. of wire projecting beyond the covering; all the electrodes were immersed to a depth of 75 millims. The battery-electrodes were placed at distances of 10 millims. from the ends of the box, so that they were 284 millims. apart.

The fixed galvanometer-electrode was placed at (152, 50); then on varying the position of the other electrode, it is found that for all positions in a plane parallel to the ends of the box, and dividing it into two equal parts, there is no deflection; but at (151, 50), *i.e.* at a distance of 1 millim. from this plane, there is a deflection of 10 divisions of the scale. Hence the plane which is equidistant from the electrodes is shown to be an equipotential surface. This is the surface at which the potential is zero.

Case 7. Plate 2. fig. 9 represents three sections of three equipotential surfaces, one through the point (50, 10), another through the point (80, 10), and a third through the point (100, 10), the battery-electrodes being placed at distances of 10 millims. from the ends of the box and 284 millims. apart.

When the battery-electrodes are so near to the ends of the box, the distribution of the electric currents, and therefore the forms of the equipotential surfaces, will not be the same as in a conductor which is unlimited in every direction; hence in comparing these experimental results with theory it is necessary to take into account the influence of the ends and sides of the box.

If we measure the distances from each point of a curve to the battery-electrodes and to the positions of their electric images due to the sides and ends of the box, and if for each point we subtract the sum of the reciprocals of the distances of the negative electrode and its images from the sum of the reciprocals of the distances of the positive electrode and its images, we get a series of numbers which differ very little from one another.

If r, r', r'' .... represent the distances from the positive electrode and its images, and r_1, r_1', r_1'' .... represent the distances from the negative electrode and its images, then $\Sigma\left(\frac{1}{r}-\frac{1}{r_1}\right)$ is very nearly constant for each curve.

In Plate 2. fig. 9 the negative electrode is so far away from every point of the curve, that all except one of the electrical images at the distant end of the box have been neglected.

In the case of the curve passing through the point (50, 10), the values of $\Sigma\left(\frac{1}{r}-\frac{1}{r_1}\right)$ for the several points are given in the following table:—

·0444
·0443
·0439
·0435
·0432
·0433

The curves are drawn only on one side of the axis; but they are symmetrical about the axis, and also, with regard to the plane of zero, potential.

Treating the next curve in the same way, the coordinates of the first point taken in this curve are 80, 10, and the battery-electrodes are at the same point as before, $\Sigma\left(\frac{1}{r}-\frac{1}{r_1}\right)$ for each point.

·0316	·0310	·0313
·0314	·0311	·0312
·0314	·0309	·0315
·0312	·0311	·0316
·0311	·0311	·0319

The mean value of $\Sigma\left(\frac{1}{r}-\frac{1}{r_1}\right)=$ ·0313, neglecting all the images but one of the more distant electrode.

These images would diminish the potential at each point, but by very nearly the same amount, as they are so far away from every point of the curve.

Another curve is drawn on this Plate through the point (100, 10). These three curves were drawn with the battery-poles in the same positions.

Case 8. In a curve (see Plate 2. fig. 10) drawn through the point (120, 20), with the electrodes at a distance of 80 millims. from the ends of the box, only the nearest electrical image was taken into account: the values of $\left(\frac{1}{r}-\frac{1}{r_1}+\frac{1}{r'}\right)$ for the several points are:—

·0183	·0183	·0173
·0181	·0183	·0170
·0183	0182	·0166
·0182	·0182	·0167
·0183	·0178	·0163
·0182	·0177	·0160
·0183	·0172	

In this case all the images except one are neglected. When the curve approaches near the side of the box, as at these latter points, it becomes necessary to take account of the effect due to the side of the box; and the forms of the curves are completely altered.

Case 9. In the experiments with sulphate of zinc (Plate 1. fig. 11), the zinc electrodes were not immersed to any considerable depth in the liquid, as in the other experiments already described, but were rounded off and immersed to a depth of about 3 or 4 millims., in order to make good contact, the galvanometer-electrodes being somewhat smaller than the others and only just dipping into the liquid. The other arrangements were the same as before. No electrical images were taken into account, as the battery-electrodes were at a distance of 102 millims. from the ends of the box. Taking the first curve described, the values of $\left(\frac{1}{r}-\frac{1}{r_1}\right)$ for the several points are:—

·0139	·0143	·0142
·0136	·0143	·0143
·0137	·0145	·0142
·0141	·0146	·0141
·0143	·0143	·0138

The highest numbers are the values for points which are nearest to the side of the box. The battery-electrodes are 100 millims. apart, and equally distant from the centre of the box. The curve is drawn through the point (180, 15).

Plate 1. fig. 11 also contains another curve for sulphate of zinc, where the electrodes only just dip into the liquid, the battery-electrodes, as before, being 100 millims. apart and equidistant from the centre of the box. The curve is drawn through the point (170, 15). Taking the 40 points which have been determined experimentally for this curve, the values of $\left(\frac{1}{r}-\frac{1}{r_1}\right)$ are contained in the following table, arranged in four columns:—

·0080	·0082	·0083	·0079
·0080	·0081	·0083	·0078
·0078	·0081	·0082	·0078
·0078	·0081	·0080	·0078
·0079	·0081	·0081	·0076
·0079	·0081	·0081	·0077
·0079	·0081	·0080	·0076
·0080	·0081	·0079	·0076
·0081	·0082	·0079	·0074
·0082	·0083	·0079	·0074

The last points taken on the curve are nearer to the side and also to the end of the box than to the nearest electrode; hence for these points corrections should be applied depending on the changes of potential due to the side and end of the box.

It also appears very clearly from this series of values that, by reversing the battery-current at each observation, the polarization of the galvanometer-electrodes may be kept small and very nearly constant throughout a long series of experiments.

Experiments with Linear Electrodes.

Case 10. The rectangular box was also employed to determine the forms of the cylindrical equipotential surfaces when the electrodes are straight rods and extend throughout the depth of the liquid. Sulphate of zinc was employed for these experiments, and the electrodes were amalgamated zinc rods. Plate 2. fig. 12 represents the sections of the equipotential surfaces when one battery-electrode is at A, the centre of the box, and the other at B, the middle point of one side.

These electrodes are 100 millims. apart. The galvanometer-electrodes in these experiments could not be brought within less than 5 millims. of the axis of the curves.

The curves drawn represent the sections of equipotential cylindrical surfaces which are at distances of 10 millims. apart, measured along a line which is parallel to and 5 millims. distant from the axis.

The case of a circular cylinder containing sulphate of copper, with the battery-electrodes at the two ends of a diameter, has also been worked out experimentally; and the equipotential surfaces are circular cylinders cutting the sides of the vessel at right angles. Mercury was tried for these experiments; but from its almost perfect conducting-power it was very difficult to determine two vertical lines in it which were precisely of the same potential.

Case 11. Another case was worked out experimentally with line-electrodes in sulphate of zinc. One positive electrode was placed at the middle point of one side of the rectangular box, and two negative electrodes were placed symmetrically at the same distance from the positive electrode; so that the lines joining the positive to the two negative electrodes were at right angles to one another. This corresponds to the case drawn in Plate 2. fig. 4; and the sections of the surfaces at points which are not near the side of the box are the same as the curves in figs. 3 or 4 of Plate 2. Having previously determined the forms of these curves in the tinfoil experiments, it was very easy to move the tracing-electrode from one point to another on the same equipotential curve, by making use of the curves previously drawn as guides for the electrode.

Comparison of Experimental results with the Theory of Electrical Distribution.

To examine the forms of the equipotential curves and surfaces with a view to determine how far the results of experiment agree with and are accounted for by the theory of distribution of electricity in a plane and in space of three dimensions, take first the case of a plane of unlimited extent in every direction, when two battery-electrodes are joined to two points in the plane, so that there are equal and opposite currents at those points. The potential at any point P of the plane is equal to $C-A(\log r-\log r_1)$, where r, r_1 are the distances of the point P from the two battery-electrodes A and B. Hence the condition for an equipotential curve is $\log r-\log r_1=\log c$, or $r=cr_1$, where c is some constant. This equation indicates that the bisector of the angle at any point P between the lines r and r_1 cuts the line joining the two battery-electrodes in the same point, which is a point on the equipotential curve.

If $2a$ is the distance between the electrodes, the distances from A and B to this intermediate point are $\frac{2ac}{1+c}$ and $\frac{2a}{1+c}$. The locus of the point P is a circle whose diameter is $\frac{4ac}{1-c^2}$, and whose centre is on AB produced.

If we refer this circle to rectangular coordinates with the axis of y equidistant from A and B, then its equation is

$$(x-a)^2+y^2=c^2\{(x+a)^2+y^2\},$$

or

$$\left(x-a\,.\,\frac{1+c^2}{1-c^2}\right)^2+y^2=\left(\frac{2ac}{1-c^2}\right)^2.$$

If

$$y=o,\ x=a\,.\,\frac{1+c}{1-c}\text{ or }a\,.\,\frac{1-c}{1+c},$$

giving the points where any equipotential curve cuts the axis.

The point B is the electrical image of A with regard to each of these circles; for if O be the centre of one of the circles,

$$\text{OA}\,.\,\text{OB}=\left(\frac{2ac}{1-c^2}\right)^2.$$

The lines of flow cut the equipotential curves at right angles, or the radius of one of these circles is the tangent to a line of flow; hence it follows (Euclid, Book III. Prop. 8) that the lines of flow are arcs of circles passing through the points A and B, the positions of the electrodes. This result may be arrived at by a method which is also applicable to the case of unequal charges at several points in a plane.

Let us take the equation to the equipotential curves in the form $(x-a)^2+y^2=c^2\{(x+a)^2+y^2\}$. Differentiating and eliminating c we get

$$\frac{(x-a)+y\dfrac{dy}{dx}}{(x-a)^2+y^2}=\frac{(x+a)+y\dfrac{dy}{dx}}{(x+a)^2+y^2}.$$

The lines of flow cut the equipotential curves at right angles; hence for a line of flow

$$\frac{(x-a)\dfrac{dy}{dx}-y}{(x-a)^2+y^2}=\frac{(x+a)\dfrac{dy}{dx}-y}{(x+a)^2+y^2}.$$

Integrating this equation, we get

$$\tan^{-1}\frac{y}{x-a}-\tan^{-1}\frac{y}{x+a}=\text{constant},$$

showing that the lines of flow are circles passing through the positions of the two electrodes. Comparing this result with the curves drawn by experiment in Plate 1. fig. 1, we see that for all parts of the disk which may be considered to be beyond the influence of the edge of the disk, there is a very close agreement between theory and experiment; and the influence of the edge of the disk will be such that the equipotential curves will cut the edge of the disk at right angles.

When the conducting-sheet is not of infinite extent, but of very considerable extent in every direction, the expression C$-$A $(\log r-\log r_1)$ will still be a close approximation to the potential at any point, provided the electrodes are not far from the centre of the sheet. Hence on such a sheet under these conditions the portions of equipotential curves and lines of flow which are not near the edges of the sheet will be very nearly circular. This we have seen to be the case in the experiments on tinfoil (see Plate 1. fig. 1).

When the lines approach near the edge of the disk, or when the electrodes are near the edge, then the form of the equipotential curve is generally altered, because the edge of the disk will generally cut some of the lines of flow of the infinite plane. We may conceive of the lines of flow as distinct from one another, but as filling up the whole of the disk; if, then, we make a section of the disk along a line of flow we do not alter the distribution of electricity on the disk, and so do not alter the forms of the equipotential curves. But in that case, since the two parts of the disk are entirely separate from one another, one of the parts may be taken away without causing any change in the law of distribution of the electric currents in the other.

If, then, we cut away from an indefinite sheet a portion which is bounded by lines of flow, we shall not alter the form of the equipotential curves; but, since we thereby increase the resistance of each portion, we shall increase the difference of potential between the two electrodes, and the distances between successive equipotential curves corresponding to given differences of potential will be diminished.

The straight line joining the battery-electrodes is a line of flow; hence when the electrodes are on one edge of a sheet which is unlimited in all directions but one, the equipotential curves are still arcs of circles having their centres on the straight edge of the disk; or if a portion of the disk bounded by this edge and by the arc of a circle which passes through the two electrodes be cut out, the forms of the equipotential curves in this portion will be arcs of circles with their centres on the straight edge.

The same will apply to any portion bounded by the arcs of two circles passing through the two electrodes; so that on a circular disk with the two electrodes on the circumference, the circular boundary of the disk will be two lines of flow, and the equipotential curves which cut them at right angles will also be circles with their centres on the straight line passing through the two electrodes*. One of the electrodes will be the electrical image of the other with regard to these circles. This has also been proved to be the case by the experiments (see Plate 2. fig. 2).

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* [*July* 23.—This case has been worked out experimentally by M. Kirchhoff by joining the galvanometer-wires to two points of the plane sheet which are at different potential, and balancing the current which would flow through those wires, by introducing a thermoelectric pile into the galvanometer-circuit. M. Verdet says: "M. Kirchhoff n'a étudié par l'expérience que le cas d'une plaque circulaire communiquant par deux points de sa circonférence avec les réophores d'une pile." M. Kirchhoff used a copper disk, and from its being a good conductor was unable to determine its resistance. M. Verdet also adds that M. Quincke worked out the case of a square disk with one battery-pole at a corner and the other on a diagonal of the square. These two are the only cases con-

If in an infinite plane there be any number of electrodes, the equipotential curves and lines of flow may be drawn by the method explained by Professor Maxwell[*], by drawing circles of equal differences of potential around each charge, and joining their point of intersection in such a way that for all the points taken the sum of the potentials is constant.

The potential at any point of the plane is $C+A\log r+B\log r'-E\log r_1-F\log r_1'+\&c.$, where r, r', r_1, r_1' are the distances of the point from the several electrodes.

In the case where the currents are equal at the several points the potential is $C+A\log\left(\frac{r\ r'\ r''\ \ldots}{r_1\ r_1'\ r_1''\ \ldots}\right)$. Hence for an equipotential curve we must have

$$r\ r'\ r''\ \ldots = c\ r_1\ r_1'\ r_1''\ \ldots,$$

or $\log r+\log r'+\log r''-\log r_1-\log r_1'-\log r_1''=\log c$.

By differentiating and finding the curves which cut these at right angles we shall find, as on p. 13, that the equation to a line of flow is of the form

$$\Sigma\left\{\tan^{-1}\left(\frac{y-b}{x-a}\right)-\tan^{-1}\left(\frac{y-b_1}{x-a_1}\right)\right\}=\text{constant},$$

or $$\Sigma\,(\theta-\theta_1)=\text{constant}.$$

When currents flow into and out of a plane conducting-sheet at several points of it, the potential at any point of the sheet is

$$C+A\log r+B\log r'-E\log r_1-F\log r_1'+\&c.,$$

where A, B, &c. are proportional to the quantity of current flowing into the sheet, and E, F, &c. to the quantity flowing out at the several battery-electrodes. In the case of battery-electrodes where there is no loss of current from the conducting-sheet, these currents may always be divided up into a series of pairs of equal and opposite currents, so that there will be an equal number of positive and negative terms. If, then,

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tained in this paper which have been previously worked out experimentally. The case worked out by M. Quincke is shown to be the same as that (Plate 2. fig. 3) of one positive electrode at the centre and four negative electrodes at the four corners of a square in the centre of a sheet of tinfoil which is so large compared with the size of the square, that it may be considered of unlimited extent. Some other cases of limited sheets, with other arrangements of electrodes so as to give the same curves, have also been considered in this paper. Since this paper was read before the Royal Society, Prof. G. C. Foster has given, in the 'Philosophical Magazine' for May and June 1875, an account of the work which had been previously done on this subject, in which he draws attention to an interesting paper on the theory of lines of flow in a plane, by Prof. W. R. Smith, published in the Proceedings of the Royal Society of Edinburgh for 1869–70.]

[*] Treatise on Electricity and Magnetism, vol. i. p. 148.

currents at the several electrodes are all equal, then for an equipotential curve we shall have

$$r\,r'\,r''\,\ldots = c\,r_1\,r_1'\,r_1''\,\ldots$$

If the current enters at one point and leaves by four currents equal to one another, then

$$r^1 = c\,r_1\,r_1'\,r_1''\,r_1'''.$$

We have an instance of this case (Plate 2. fig. 3) when a current enters a very large sheet of tinfoil at the centre, and leaves by the four corners of a square round this electrode as a centre.

When a current enters a conducting-sheet by one point and leaves in equal currents by two others, the equation to an equipotential curve becomes

$$r^2 = c\,r_1\,r_1',$$

where r is the distance from the positive electrode, and r_1, r_1' the distances from the negative electrodes to any point of the curve.

In the particular case when $c = 1$, and when the electrodes are all in the same straight line, this becomes a rectangular hyperbola, the foci of which are the positions of the negative electrodes, and the positive electrode is at the centre of the hyperbola.

This is the case which is given in fig. 5, as worked out experimentally, where the focal distance of the hyperbola is 76 millims., and its vertex is at a distance from the centre equal to 53·75 millims., *i. e.* very nearly $\frac{1}{\sqrt{2}} \times 76$ millims. The measured values of $r_1 - r_1'$, starting from the axis for six successive points some millims. apart from one another, are 108, 109, 108, 108, 108, 108 millims. The theoretical value for this difference is 107·5 millims.

The Theory of Electrical Images.

When there are four equal electrodes with currents entering the sheet by two of them and leaving by the other two, we have

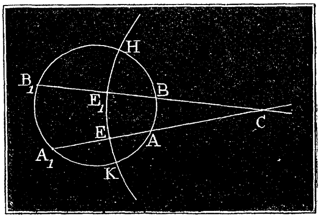
$$rr' = c\,r_1\,r_1'.$$

Consider the case when these four electrodes lie on the circumference of a circle. Join each pair of like electrodes by straight lines, and produce the chords of the circle so joined to meet one another.

Let AA_1, BB_1 be these chords meeting in C. Then the potential at the point C is

$$c + A\{\log(CA\,.\,CA_1) - \log(CB\,.\,CB_1)\} = c.$$

Fig. 1.



Draw round C as centre the circle which cuts the circumference of $A A_1 B_1 B$ at right angles, and let it cut the circle in H and K, and the chords in E and E_1.

$$\text{Then } CA \,.\, CA_1 = CE^2 = CK^2,$$
$$\text{and } \quad CB \,.\, CB_1 = CE^2_1 = CH^2.$$

If the value of the constant c is zero in the above expression, then the centre C is a point of zero potential.

If we produce the other two chords of the circle to meet in the point O, and take a point D in AB, such that $OA \,.\, OB = OD^2$, and draw a circle about O with OD as radius, then this circle is an equipotential curve for the four electrodes A, A_1, B, B_1. For if P be any point of this circle, then

$$PB : PA :: DB : DA,$$

and

$$PB_1 : PA_1 :: D_1B_1 : D_1A_1 :: DB_1 : DA_1.$$

But the potential at any point, P, due to the four electrodes is

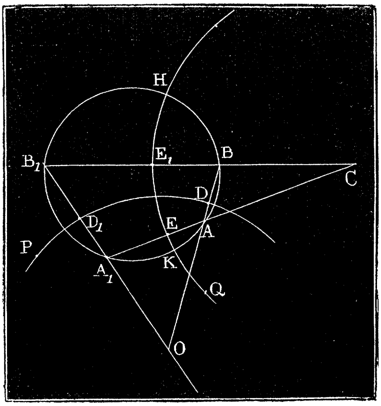
$$A \log \left(\frac{PA \,.\, PA_1}{PB \,.\, PB_1}\right);$$

hence on the circle DD_1 this becomes

$$A \log \left(\frac{DA \,.\, DA_1}{DB \,.\, DB_1}\right),$$

which is the same as the potential at the point D.

Fig. 2.



Since

$$DA : DB :: OD : OB,$$

and

$$D_1A_1 : D_1B_1 :: OA_1 : OD_1, \text{ or } :: OD_1 : OB_1,$$

the potential at D is

$$A \log \left(\frac{OA_1}{OB}\right), \text{ or } A \log \left(\frac{OA}{OB_1}\right), \text{ or } A \log \left(\frac{OD^2}{OB \,.\, OB_1}\right).$$

These results show that the circle through DD_1 is an equipotential curve for the four equal electrodes A, A_1, B, B_1; also that the potential at any point of this circle is the same as the potential at the centre O would be if there were only a positive electrode at A and a negative electrode at B_1, which is the same as one half the potential at the centre, O, due to the four electrodes.

If we take two of these electrodes, B, B_1, and through C draw another chord through two other electrodes on the circle, and draw the equipotential circle for this set of four electrodes, we shall find that all such circles cut both the circles AA_1B and EE_1 mutually at right angles, and that the circle EE_1 is a line of flow for all such systems of four electrodes on the circle AA_1, B_1B.

The circles DD_1, EE_1, and AA_1B will intersect mutually at right angles.

This will appear if it is shown that AA_1 and BB_1 intersect on the common chord of the circles DD_1 and AA_1B; for in that case the tangent to DD_1 drawn from C is equal to the radius of EE_1.

Let the circle AA_1B be referred to two tangents through the point C and their common chord ($\gamma=0$); then its equation is $\alpha\beta-\gamma^2=0$.

Let $\mu^2\alpha-\beta=0$ and $\lambda^2\alpha-\beta=0$ be the equations to the two lines CAA_1 and CBB_1.

We may call the points A, A_1, B, B_1, the points μ, $-\mu$, λ, $-\lambda$. The equation to the chord joining μ and λ is

$$\lambda\mu\alpha-(\lambda+\mu)\ \gamma+\beta=0,$$

and the equation to the chord joining $-\mu$ and $-\lambda$ is

$$\lambda\mu\alpha+(\lambda+\mu)\ \gamma+\beta=0.$$

These intersect on the line $\gamma=0$ where it meets $\lambda\mu\alpha+\beta=0$.

The equations to the tangents through this point are

$$\lambda\mu\alpha-2\sqrt{\lambda\mu}\,.\,\gamma+\beta=0$$

and

$$\lambda\mu\alpha+2\sqrt{\lambda\mu}\,.\,\gamma+\beta=0,$$

which touch the circles at the points $\sqrt{\lambda\mu}$ and $-\sqrt{\lambda\mu}$.

The equation to the chord joining these points is

$$\lambda\mu\alpha-\beta=0,$$

which passes through the point (α, β).

The circle HEK is a line of flow for any such system of four equal electrodes as A, A_1, B, B_1, lying on a circle which cuts the circle HEK at right angles.

A particular case of this arises when two of the electrodes of the same kind as A and A_1 approach to and ultimately coincide with K; then the current at K is double each of the currents at B and B_1. The two lines of flow, HEK and KBB_1, intersect at right angles at the point K, and the equipotential circle DD_1 diminishes to a point.

Since the circle HEK is a line of flow, we may cut the conducting sheet along the arc of this circle without changing the form of the equipotential curves. If, then, at two points A and B, on a circular disk HEK, we have two equal and opposite battery-electrodes, the equipotential curves will be the same as the equipotential curves in an infinite sheet which are due to those two electrodes, together with equal and like electrodes at their electrical images with regard to the circle*.

Extension of the Theory of Electrical Images.

We have seen that when the boundary-line of a disk is made up of arcs of circles passing through the two battery-electrodes, then the forms of the lines of flow will be the same as in an infinite sheet; and consequently the forms of the equipotential curves are arcs of circles.

We have also seen that when the boundary of the disk is a circle, and the two electrodes lie within it, the forms of the lines of flow will be altered because of the presence of the edge of the disk; and the change in the electrical distribution will be that due to equal quantities of electricity in the position of the electrical images of the electrodes formed by the edges of the disk. If from a circular disk with two equal battery-electrodes lying within it, we cut off a portion bounded by the arc of that

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* [*July* 23.—That the circle HEK is a line of flow for the four electrodes may be readily seen if we resolve the rate of flow at any point, Q, of the circumference of this circle in the direction of the radius at that point. As shown by Prof. W. R. Smith, the rate of flow arising from each electrode is inversely as the distance from the electrode.

Let $CA=a_1$, $CA_1=a_1$, $AQ=r$, $A_1Q=r'$, and ρ the radius; then, according to Prof. Smith's notation, the resolved part along the radius from the two like electrodes A and A_1 will be

$$\frac{E}{2\pi r}\times\frac{r^2+\rho^2-a^2}{2r\rho}+\frac{E}{2\pi r'}\times\frac{r'^2+\rho^2-a_1{}^2}{2r'\rho},$$

or

$$\frac{E}{2\pi\rho}+\frac{E}{4\pi\rho}\left\{\frac{\rho^2-a^2}{r^2}+\frac{\rho^2-a_1{}^2}{r'^2}\right\};$$

but

$$\frac{r}{r'}=\frac{\rho-a}{a_1-\rho}=\frac{\rho+a}{a_1+\rho},$$

so that

$$\frac{r^2}{r'^2}=\frac{\rho^2-a^2}{a_1{}^2-\rho^2}\text{; therefore }\frac{\rho^2-a^2}{r^2}+\frac{\rho^2-a_1{}^2}{r'^2}=0.$$

Hence the rate of flow along the radius, viz. $\frac{E}{2\pi\rho}$, is the same as if there were an equal electrode at C.

In the same way it may be shown that the rate of flow along the radius arising from the other two electrodes is the same in quantity, but in the opposite direction.

Hence there is no flow along the radius at the point Q when there are two equal positive electrodes at A and A_1, and two other equal negative electrodes at B and B_1.

So that at every point, Q, of the circle HEK the circumference of the circle must be the line of flow.]

circle which passes through the two electrodes and cuts the circular disk at right angles, we shall not alter the forms of the equipotential curves within either of the separate disks bounded by the arcs of these two circles.

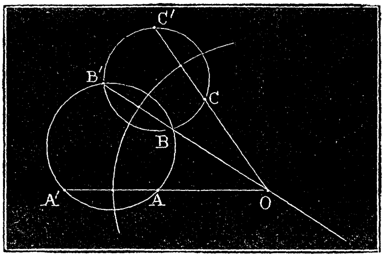
In all these cases the boundary-line will be made up of lines of flow, and the equipotential curves will *generally* cut the edge of the disk at right angles. Or if, in an unlimited sheet, each of two battery-electrodes be divided into two, so that equal currents flow through each point, and if these four electrodes be so arranged that they all lie on the circumference of a circle, then the forms of the equipotential curves will be the same as in the case of a circular disk which has its centre at the point of intersection of the chords joining the two positive electrodes and the two negative electrodes respectively, and whose radius is equal to the tangent drawn from this point to the circle on which the electrodes lie. In the case of the circular disk, the positive electrode outside the disk becomes the virtual electric image of the other positive electrode, and the outside negative electrode becomes the electric image of the other negative electrode. The edge of the disk cuts the other circle at right angles, and is a line of flow for the four equal electrodes in the unlimited sheet. Hence no alteration is made in the forms of the lines of flow by cutting the sheet along this edge.

This will apply to the case of any two unlike equal electrodes within a circular disk when there are any number of electrodes connected with the disk; for if we take all these electrodes and their electric images, and suppose them to be kept continually charged, each image like its point, then in the unlimited sheet the line in the position of the edge of the disk will be a line of flow for every set of two unlike electrodes and their images, and therefore for all of them when taken together.

Hence, by cutting along the edge of the circular disk to which such images are due, no change is made in the lines of flow or in the equipotential curves. In fig. 3, A and C are like electrodes.

When the currents entering or leaving the disk at the several electrodes are not all equal, since the sum of the currents entering the disk must be equal to those leaving it, they may all be divided up into sets of

Fig. 3.



two unlike equal currents; and hence for all cases of currents entering and leaving a circular disk, the edge is in the position of a line of flow for all the electrodes and their images in the unlimited sheet. At an electrode and its image the currents are equal, and must be supposed to be both entering or both leaving the sheet.

In the case when the two equal sets of two unlike electrodes in the unlimited sheet lie at the corners of a rectangle, then the straight lines joining the like poles are bisected at right angles by a straight line, which is a line of flow.

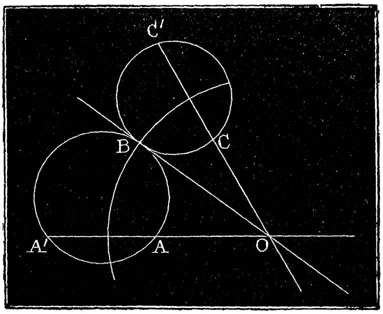
Hence, if the sheet be cut along this straight line, no alteration is made in the lines of flow on either side of it; and the pair of unlike electrodes on one side of it become the images of the electrodes on the other side.

In the case of a plane sheet limited by two lines at right angles, three electrical images of each electrode will be formed by the two rectangular intersecting sides; and these images will be arranged in a circle about the point of intersection, in the same positions as the optical images formed by two plane mirrors at right angles to one another.

The equipotential curves in this case will be the same as those in an infinite sheet when there are four equal electrodes of each kind—the four positive electrodes taking the positions of the one positive electrode and its three images, and the four negative electrodes being similarly placed.

A particular and an interesting case arises when one of the electrodes is on the edge of the circular disk; the electric image of this electrode then coincides with the electrode itself; and the case in an unlimited sheet which corresponds to this is that where a current enters the sheet at one point, and leaves it by equal currents at two other points, the three points lying on a circle which touches the radius of the disk, *i. e.* cuts the disk at right angles, at the point where the current enters the sheet.

Fig. 4.

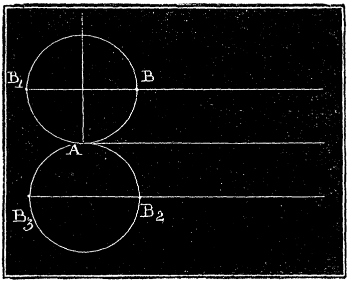


When a current enters a circular disk at one point on the edge, and leaves it by equal currents at two other points, the corresponding case in an unlimited sheet is that where a current enters the sheet at one point and leaves by equal currents at four other points lying two and two on circles which touch one another at the point where the current enters the sheet (fig. 4).

If, in an indefinite sheet (Plate 2. fig. 3), one of the battery-electrodes be separated into four parts and attached to the tinfoil at the four corners of a square, and if the other battery-electrode be attached at the centre of the square, then the current at the centre will be four times the current at each corner of the square; and the forms of the equipotential curves will be the same as when there are only two, unlike equal electrodes in a sheet limited in two directions by straight lines meeting at the centre and parallel to the sides of the square, the three electrical images in the limited sheet being in the same positions as three of the electrodes in the unlimited sheet.

This is a particular case of the last, when an electrode and its image lie at the ends of a diameter of the circle, and are equally distant from the point at which the current enters the sheet (fig. 5).

Fig. 5.



The case with one positive electrode at the centre and four negative electrodes at the corners of a square in an unlimited sheet may also be regarded as equivalent to two sets of electrodes, each set consisting of one positive electrode at the corner of a square and two negative electrodes at equal distances along the two edges measured from that corner when the sheet is cut along these two edges.

The curves are also the same as in a sheet limited by two edges at right angles to one another, when one electrode is at the corner and the other on the diagonal of the square sheet, the electrodes being at the same distance as in the two previous cases.

The curves in Plate 2. fig. 4 are drawn on such a sheet with one positive electrode at A and a negative electrode at B, at a distance of 3 inches from A, the whole sheet of tinfoil being 18 inches square.

It may also be regarded as a particular case of two sets of electrodes in an unlimited sheet, where there are three electrodes in each set at equal distances along the same straight line, the current entering at the middle one and flowing out at the other two electrodes. Since the circle is not a line of flow for such a system of electrodes, we cannot have this system of equipotential curves and lines of flow in a circular disk.

A particular solution of this case, the equipotential curve for zero potential, is a rectangular hyperbola, having its centre and two foci at the three electrodes (see Plate 1. fig. 5).

In the case of a circular disk the electric image of the centre may be regarded as a point at an infinite distance, so that the distance of the image from any point of the circular disk remains constant.

If, then, one electrode be at the centre and the other at the circumference of a circular disk (Plate 1. fig. 6), the corresponding case in the unlimited sheet is that where the current flowing in at the circumference is double the current flowing out at the centre, we may conceive of it as a case where there is one electrode at an infinite distance, equal to and of like kind with the electrode at the centre.

In such a case the potential at any point is expressed by $C+A\log r-2A\log r_1$

Hence for an equipotential curve

$$r_1^2=2\mu ar,$$

where a is the radius of the circle (which is introduced for the sake of making the equation homogeneous), and μ is some ratio which is constant for the same curve.

With polar coordinates about the centre, taking a as the radius of the circle, we get

$$r^2+a^2-2ar\cos\theta=2\mu ar,$$
$$r^2-2ar(\mu+\cos\theta)+a^2=0.\ .\ .\ .$$

If $\mu=$ or >2, the curve is a closed curve, and, when r is small compared with a, the curve nearly coincides with an ellipse of eccentricity $\frac{1}{\mu}$ and mean distance $\frac{\frac{1}{2}\mu a}{\mu^2-1}$:

$$r=\frac{a^2+r^2}{2a\mu\left(1+\frac{1}{\mu}\cos\theta\right)}.$$

Hence $\frac{r^2}{a^2}$ expresses the percentage of error in the value of r if the ellipse be taken instead of the curve.

In a circular disk of 100 millims. in radius the distance between the curve and the ellipse at a distance of 25 millims. from the centre of the disk is only about 1·5 millim.

When μ is less than 2 and greater than $(\sqrt{2}-1)$, the curves are of double curvature, and have points of inflection.

From the equation of the curve we obtain

$$\{r-a(\mu+\cos\theta)\}\frac{dr}{d\theta}+ar\sin\theta=0.$$

If $r=a$, then $\mu+\cos\theta=1$.

Hence $r\,.\,\frac{d\theta}{dr}=0$; and the curves generally cut the edge of the disk at right angles.

When $\mu=2$ and $\cos\theta=-1$, then the two branches of the curve meet on the edge at the extremity of the diameter through the two poles.

On this curve

$$r\,.\,\frac{d\theta}{dr}=\frac{\sqrt{(2+\cos\theta)^2-1}}{\sin\theta}=\frac{\sqrt{(1+\cos\theta)(3+\cos\theta)}}{\sin\theta}=\sqrt{\frac{3+\cos\theta}{1-\cos\theta}}.$$

Hence when $\cos\theta=-1$,

$$r\frac{d\theta}{dr}=\pm 1,$$

and the two branches of the curve are each inclined at an angle of 45° to the edge of the disk.

The polar equation to this curve is very much simplified when this double point is taken as the origin of coordinates.

The polar equation then becomes

$$r^2+4a^2-4ar\cos\theta=4a\sqrt{r^2+a^2-2ar\cos\theta}\,.\,r^4-8ar^3\cos\theta$$
$$+16a^2r^2\cos^2\theta+8a^2r^2=16a^2r^2\,.\,(r-4a\cos\theta)^2$$
$$=8a^2\,.\,r=4a\cos\theta\pm 2\sqrt{2}-a;$$

and $\frac{dr}{d\theta}=-4a\sin\theta$.

From this result it is easy to see that there is no point of inflection in the curve; so that each branch of the curve is continuous and remains on the same side of its tangent.

From the above equations the curve may be traced. It consists of two loops, which form a continuous curve, the two branches cutting one another at right angles in passing through the double point on the edge of the disk.

The complete equipotential curves for the unlimited sheet may also be readily traced from the above equations to those curves by giving different values to μ (see Plate 1. fig. 7).

Neglecting the part of the curve outside the disk, the equation to the curve may be put under the form

$$\frac{\{r-a(\mu+\cos\theta)\}}{a}=-\sqrt{(\mu-1+\cos\theta)(\mu+1+\cos\theta)}.$$

To find the points where it cuts the axis, let $\cos\theta=\pm 1$; then

$$\frac{r}{a}=(\mu\pm 1)-\sqrt{\mu(\mu\pm 2)},$$

where the upper signs are to be taken together.

In the case when $\mu = 2$,

$$\frac{r}{a} = 3 - 2\sqrt{2} \text{ and } 1,$$

the distances being measured in opposite directions.

To find the change in the value of $\frac{r}{a}$ when μ has a slightly different value. Let

$$\mu = 2 + \frac{1}{30} \text{ suppose ;}$$

then

$$\frac{r}{a} = 3 + \frac{1}{30} - \sqrt{\left(2 + \frac{1}{30}\right)\left(4 + \frac{1}{30}\right)}$$

$$= 3 - 2\sqrt{2} + \frac{1}{30} - \frac{2\sqrt{2}}{80} = 3 - 2\sqrt{2} - \cdot 002 \text{ nearly,}$$

or

$$= 1 + \frac{1}{30} - \sqrt{\left(2 + \frac{1}{30}\right) \cdot \frac{1}{30}} = 1 - \frac{\sqrt{61} - 1}{30} = 1 - \cdot 227 \text{ nearly.}$$

Hence if μ be increased by $\frac{1}{30}$, the curve cuts the axis at a distance of nearly one fourth of the radius from the singular point.

If μ is less than 2, the curve does not cut the axis on the negative side of the origin.

We may find where it cuts the edge of the disk by making $\frac{r}{a} = 1$.

Let $\mu = 2 - \frac{1}{30}$;

then

$$\left\{1 - 2 + \frac{1}{30} - \cos\theta\right\} = \sqrt{\left(1 - \frac{1}{30} + \cos\theta\right)\left(3 - \frac{1}{30} + \cos\theta\right)},$$

and

$$1 - \frac{1}{30} + \cos\theta = 0;$$

so that θ is very nearly 15°.

Hence a very small variation in the value of μ produces very great changes in those parts of the curve which are not near to the centre of the disk.

This is a point of very great importance in connexion with the tracing of equipotential curves on tinfoil or copper disks; and the uncertainty in the experimental determination of points in the neighbourhood of the singular point (see Case 6, p. 6) is entirely explained.

Supposing that one of the galvanometer-electrodes is placed on the straight line joining the two battery-electrodes at the point L in Plate 2. fig. 6. When the pin or electrode is 1 millim. in diameter the equipotential curve corresponding to one extremity of the diameter of the pin cuts the edge of the circular disk at a point 22° distant from the singular point, whilst the equipotential curve through the other extremity of the same diameter of the pin cuts the radius drawn to the singular point at a distance of more than one third of the radius from the singular point.

In order to draw the curves near the singular point correctly, both galvanometer-electrodes must be placed in contact with the disk in the neighbourhood of the singular point.

Electrical Distribution in Space of three dimensions.

To consider the laws of distribution of electricity in space of three dimensions, let a point or small sphere be charged with electricity; then, by Laplace's equation, the potential at any point at distance r from it is of the form $\frac{A}{r}$.

If there be two points with charges of electricity proportional to A and B, then the potential at any point is of the form $\frac{A}{r}+\frac{B}{r'}$, where r, r' are the distances from the point to the two charged spheres.

When the charges on the two spheres are equal, but of opposite kinds, the expression for the potential becomes $A\left(\frac{1}{r}-\frac{1}{r_1}\right)$.

The equipotential surfaces will then be surfaces of revolution about the straight line joining the two charged spheres. For an equipotential curve we shall have $\frac{1}{r}-\frac{1}{r_1}=\frac{1}{c}$.

On referring to the experiments with battery-electrodes immersed to a considerable depth in a conducting liquid, so as not to be near the boundary of the liquid, we see that the same equation is true for the equipotential surfaces when a steady current flows in at one electrode and out at the other.

The equation to an equipotential surface will be of the form

$$\frac{1}{r}-\frac{1}{r_1}=\frac{1}{c}.$$

Take a section of the surface through the axis, and refer the section to rectangular coordinates.

Let the axis of revolution be the axis of x, and a line equidistant from the electrodes be the axis of y, then the equation to the curve is

$$\{(x-a)^2+y^2\}^{-\frac{1}{2}}-\{(x+a)^2+y^2\}^{-\frac{1}{2}}=\frac{1}{c},$$

where $2a$ is the distance between the electrodes.

For a consecutive point in the same curve, we get

$$\frac{(x-a)\delta x+y\delta y}{r^3}-\frac{(x+a)\delta x+y\delta y}{r_1^3}=0,$$

$$\text{or } \left\{\frac{x-a}{r^3}-\frac{x+a}{r_1^3}\right\}\delta x+y\delta y\left(\frac{1}{r^3}-\frac{1}{r_1^3}\right)=0.$$

Hence for the line of flow through (xy) which will cut this curve at right angles,

$$\left(\frac{x-a}{r^3}-\frac{x+a}{r_1^3}\right)\delta y-\delta x\left(\frac{y}{r^3}-\frac{y}{r_1^3}\right)=0,$$

or

$$\frac{y^2\delta x-(x+a)y\delta y}{r_1^3}-\frac{y^2\delta x-(x-a)y\delta y}{r^3}=0,$$

or

$$\frac{\delta x}{r_1}-\frac{(x+a)\{(x+a)\delta x+y\delta y\}}{r_1^3}-\frac{\delta x}{r}+\frac{(x-a)\{(x-a)\delta x+y\delta y\}}{r^3}=0.$$

Integrating, we get

$$\frac{x+a}{r_1}-\frac{x-a}{r}=m, \text{ where } m \text{ is constant.}$$

The curve cuts the axis of y at a point where

$$\frac{a}{r}=\frac{a}{r_1}=\frac{m}{2}.$$

If α is the angle which the two radii drawn to the same point make with the axis when they are equal to one another, then $m=2\cos\alpha$.

If ϕ and θ be the angles which the two radii drawn to any point in the curve make with the axis, then the equation to a line of flow may be written in the form

$$\cos\phi-\cos\theta=2\cos\alpha.$$

This will be the equation to the line of flow in any plane through the two electrodes.

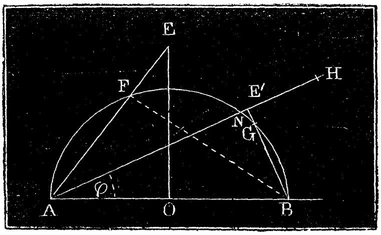
These lines of flow coincide with the lines of force in non-conducting space with equal and opposite charges in the positions of the two electrodes. They also coincide with the magnetic lines of force when the two poles are in the same positions as the electrodes. From this equation the form of the lines of flow may be readily drawn.

Let AB be the two electrodes, O the middle point of AB, and OE bisect AB at right angles.

Let E be a point on the line of force. Describe a semicircle on AB cutting AE in the point F.

Take AH equal to twice AF. Place AH in any position ANH cutting the circle in N, and take a chord BG equal to NH; BG produced will meet AH on the line of force.

Fig. 6.



For $$\frac{AN}{AB}=\cos\phi, \text{ and } \frac{BG}{AB}=\cos(\pi-\theta);$$

and $$\frac{AN}{AB}+\frac{BG}{AB}=\frac{AH}{AB}=2.\frac{AE}{AB};$$

or $$\cos\phi-\cos\theta=2\cos\alpha.$$

We may regard the conducting-mass as divided into two parts by a plane passing through the axis of revolution of the lines of flow. The equipotential surfaces will all cut every such plane at right angles.

Hence we may suppose the conducting-mass on one side of such a plane to be removed without changing the form of the equipotential surfaces. The resistance of the conductor between the two electrodes will then be doubled.

Hence in a liquid the form of the lines of flow and equipotential surfaces will be the same when the electrodes are both in the surface of the liquid as when they are immersed to infinitely great depths in it. For the same reason, if the liquid be contained in a very large rectangular vessel with two plane sides parallel to the line joining the electrodes, the forms of the equipotential surfaces will be the same when the electrodes are on the line where the surface of the liquid meets one side of the box, provided they are not near the ends of the box, and provided the opposite side of the box is too far off to disturb the forms of the equipotential surfaces.

A particular solution of the equation to the lines of flow is

$$\cos\theta-\cos\phi=1.$$

At one of the electrodes $\cos\phi=-1$, and hence $\theta=\frac{\pi}{2}$; and at the other $\cos\theta=1$, and hence $\phi=\frac{\pi}{2}$; so that the lines of flow cut the axis at right angles, *i. e.* the lines of flow given by this equation touch the two planes drawn through the electrodes at right angles to the axis.

In order that the lines of flow in a vessel containing liquid may coincide accurately with the theoretical lines of flow in a liquid of infinite extent in all directions, it is necessary that every section of the boundary of the vessel through the electrodes should be a line of flow.

From the above considerations, it appears that near the electrodes the ends of the vessel may be plane and at right angles to the axis, without causing much change of form in the equipotential surfaces.

When there are charges of electricity at several points, then the potential at any other point is the algebraic sum of the potentials at that point due to the several charges separately.

This will also be the case in a liquid conductor of unlimited extent in every direction when currents enter and leave the liquid at several points at a great depth within it.

The potential at any point is of the form

$$\frac{e}{r}-\frac{e_1}{r_1}+\frac{e_2}{r_2}-\frac{e_3}{r_3}+\&c.$$

If the currents flowing out and in at the several points are all equal, then the potential is of the form

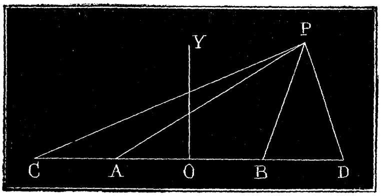
$$A\left(\frac{1}{r}-\frac{1}{r_1}+\frac{1}{r_2}-\frac{1}{r_3}+\&c\right).$$

If the electrodes are all in one plane, then this plane may be taken as one of the bounding surfaces of the conducting liquid, without altering the forms of the equipotential surfaces, which cut the plane always at right angles.

If r, r_1, r_2, r_3 are the distances of a point from the electrodes, then, with four electrodes, the equation to an equipotential surface will be

$$\frac{1}{r}-\frac{1}{r_1}+\frac{1}{r_2}-\frac{1}{r_3}=\frac{1}{c}.$$

Fig. 7.



Suppose the electrodes to be in the same straight line at A, B, C, D; and let $AB=2a$, and $CD=2b$; the positive electrodes at A and C, and the negative electrodes at B and D; then the equation to a section of an equipotential surface through the axis referred to rectangular coordinates, as before, will be

$$\{(x-a)^2+y^2\}^{-\frac{1}{2}}-\{(x+a)^2+y^2\}^{-\frac{1}{2}}+\{(x-b)^2+y^2\}^{-\frac{1}{2}}-\{(x+b)^2+y^2\}^{-\frac{1}{2}}=\frac{1}{c_1}.$$

Differentiating, we get for a consecutive point

$$\frac{x-a+y\frac{dy}{dx}}{\{(x-a)^2+y^2\}^{\frac{3}{2}}}-\frac{(x+a)+y\frac{dy}{dx}}{\{(x+a)^2+y^2\}^{\frac{3}{2}}}+\&c=0.$$

Hence for a line of flow which cuts this at right angles,

$$\frac{(x-a)\frac{dy}{dx}-y}{\{(x-a)^2+y^2\}^{\frac{3}{2}}}-\frac{(x+a)\frac{dy}{dx}-y}{\{(x+a)^2+y^2\}^{\frac{3}{2}}}+\&c.=0.$$

Hence, multiplying by y and integrating as before, we get

$$\frac{x+a}{r_1}-\frac{x-a}{r}+\frac{x+b}{r_3}-\frac{x-b}{r_2}=c,$$

or

$$\cos\theta-\cos\phi+\cos\theta_1-\cos\phi_1=c,$$

where θ, θ_1, ϕ, ϕ_1 are the angles which the radii make with the line on which the electrodes lie.

If we examine the equation

$$\cos\theta+\cos\theta_1-\cos\phi-\cos\phi_1=c,$$

which gives the lines of flow where four electrodes are arranged symmetrically along a straight line, we see that when c is greater than 2, the curve does not cut either of the two planes given by the equations

$$\cos\theta+\cos\theta_1=0, \text{ or } \cos\phi+\cos\phi_1=0,$$

but lies entirely between them, and that it cuts the axis at the two interior electrodes. The curve

$$\cos\theta+\cos\theta_1-\cos\phi-\cos\phi_1=2$$

meets the plane $\cos\theta+\cos\theta_1=0$ where it cuts the axis; for $\cos\phi=-1$, and $\cos\phi_1=-1$ for this point.

When $\cos\theta+\cos\theta_1-\cos\phi-\cos\phi_1$ is less than 2, the curve cuts the planes $\cos\theta+\cos\theta_1=0$ and $\cos\phi+\cos\phi_1=0$ at some distance from the axis, and cuts the axis at the point from which θ_1 is measured, *i. e.* it cuts the axis at the two exterior electrodes.

The curve $\cos\theta+\cos\theta_1-\cos\phi-\cos\phi_1=2$ has very close contact with the plane $\cos\theta+\cos\theta_1=0$, since the distance between them depends on the differences of small quantities of the second order. Hence in a rectangular box, when the two electrodes are near to, but not in contact with the ends, the lines of flow and equipotential surfaces should very closely coincide with the lines of flow and equipotential surfaces in space, supposing that two additional like equal electrodes are placed at the electric images of the electrodes. (See Cases 7 and 8, on pp. 9 and 10.)

The value of the constant, c, in the above equation will determine directly the cosine of the angle at which the curve cuts the axis.

For at the axis $\cos\phi=-1$, $\cos\phi_1=-1$, and *either* $\cos\theta=-1$ when c is less than 2, in which case $\cos\theta_1=c-1$, and the curve cuts the axis at the exterior electrodes at an angle $\cos^{-1}(c-1)$,
or $\cos\theta_1=1$ when c is greater than 2, in which case $\cos\theta=c-3$, and the curve cuts the axis at the interior electrodes at an angle

$$\cos^{-1}(c-3).$$

As in the case of two electrodes, if any two planes, both intersecting along the line of the electrodes, be taken as bounding surfaces of the conducting liquid, there will be no change in the forms of the lines of flow and the equipotential surfaces. The resistance of the liquid will be inversely proportional to the angle between two such bounding planes.

A case of this kind occurs when the electrodes lie on the straight line where the surface of the liquid meets the plane vertical side of the vessel containing it.

Electrical Distribution with Linear Electrodes.

Next consider the case in which the electrodes are parallel straight lines, extending throughout the liquid. With one electrode, the potential at any point at a distance r from it is

$$C+A \log r.$$

When there are several electrodes, the potential is

$$C+A \log r+B \log r_1+\&c.$$

If the currents flowing into the liquid at the several electrodes are all equal, the potential is

$$C+A (\log r+\log r_1+ \&c);$$

and for an equipotential surface we have

$$r\, r'\, r'' \ldots = c\, r_{,}\, r_{,}'\, r_{,}'' \ldots$$

where dashes below the letter indicate negative electrodes.

This equation is the same as that obtained for a plane conductor of small thickness, which has been already considered; and the forms of the equipotential surfaces are cylindrical. The disk of tinfoil may be considered as a particular case of the solid with electrodes throughout the whole thickness when that thickness is supposed to be small.

For two linear electrodes with equal and opposite currents, the equipotential surfaces will be circular cylinders, and the lines of flow will also be circular cylinders cutting them at right angles.

The particular cases of circular and other disks which have already been considered will give the sections of the equipotential surfaces for the corresponding cases with linear electrodes in the liquid, and will completely determine them.

Supposing the surface of the liquid to be of unlimited extent, and that a current enters by one line electrode, and leaves by equal currents at two negative electrodes, the forms of the cylindrical surfaces will be given by the equation

$$r^2 = cr_1 r_2.$$

When the positive electrode is at the middle point of one side of the rectangular box used in the experiments, and an equal negative electrode is within the box and near the same side, as in Case 10, the forms of the equipotential surfaces will be given by the equation

$$r^2 = cr_1 r_2;$$

r_1, r_2 are the distances of the point from the negative electrode and from its image.

This equation will give the form of the surfaces accurately only when the opposite side and the two ends of the box are far enough off that they may be regarded as practically at an infinite distance from the electrodes.

When $c=1$, the equation becomes $r^2=r_1 r_2$: so that the section of the cylindrical surface is a rectangular hyperbola.

A curve determined by experiment is drawn in Plate 2. fig. 12, very near to the position of this hyperbola; but as one of the electrodes was at the centre of the box, the forms of the equipotential curves will be slightly changed, because of the influence of the opposite side and ends of the box.

At the vertex the two curves coincide, and, measured along the latus rectum of the hyperbola, the distance between them is only 5 millims. Since the distance between the battery-electrodes is one half the breadth of the box, the influence of the opposite side and ends of the box will be quite sufficient to account for so small a difference. Thus we see that with linear electrodes there is a very satisfactory agreement between theory and experiment.

The equipotential curves for Case 10, with linear electrodes in the liquid, are theoretically the same as those for Case 5, with points on a sheet of tinfoil for electrodes. A comparison of the figures (Plate 1. fig. 5 and Plate 2. fig. 12) which are laid down from the experimental results shows how closely they agree, and may be taken to be an independent proof of the accuracy of the theory of electrical distribution.

November 18, 1875.

Dr. J. DALTON HOOKER, C.B., President, in the Chair.

In pursuance of the Statutes, notice of the ensuing Anniversary Meeting was given from the Chair.

Mr. Etheridge, Mr. F. Galton, Mr. Newmarch, Dr. W. Pole, and Mr. W. W. Smyth, having been nominated by the President, were elected by ballot Auditors of the Treasurer's Accounts on the part of the Society.

Mr. Robert Lewis John Ellery (elected 1873), Mr. James Geikie, Dr. Emanuel Klein, Prof. E. Ray Lankester, Mr. Robert Stirling Newall, and the Earl of Carnarvon were admitted into the Society.

The following extract from the Will of the late Sir Charles Wheatstone, F.R.S., in a letter from Mr. Robert Sabine, one of the executors, was read:—"I leave to the Royal Society of London for the Promotion of Natural Knowledge the Portrait of the Hon. Robert Boyle, and all the framed Portraits of scientific men now in my possession; and I also bequeath to the said Royal Society the sum of Five Hundred Pounds, to be applied to the purposes of the Wollaston Donation Fund."

W. G. Adams.

Fig 5

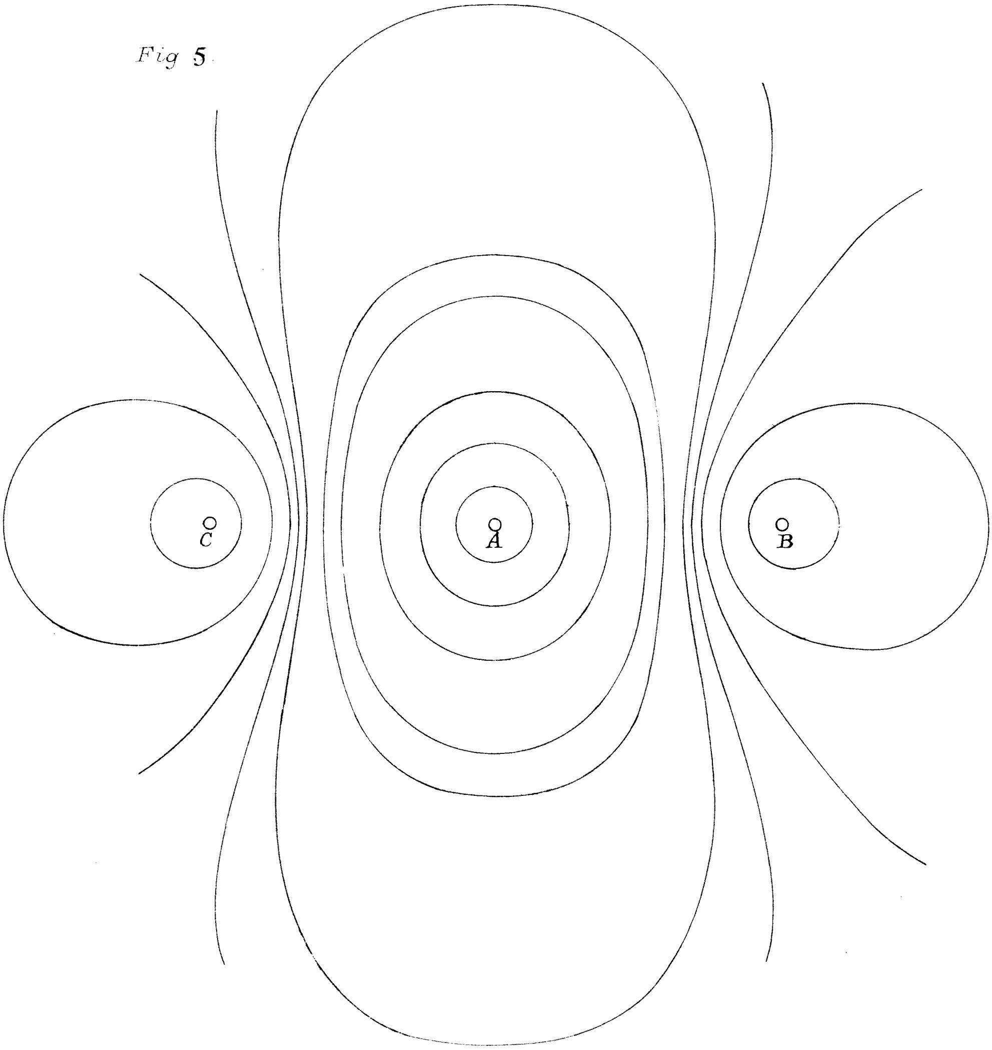


Fig. 8.

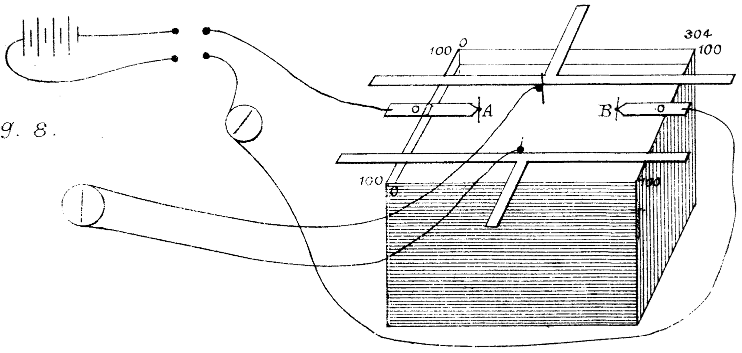
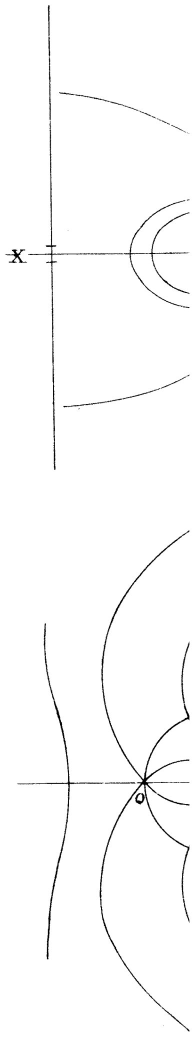
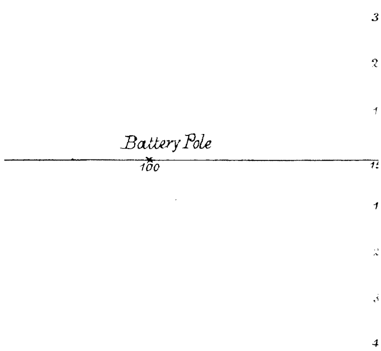


Fig.





W. H. Wesley lith.

Fig. 1

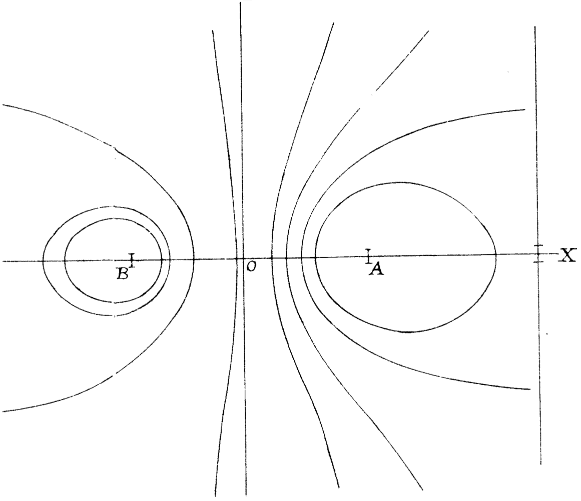


Fig. 7.

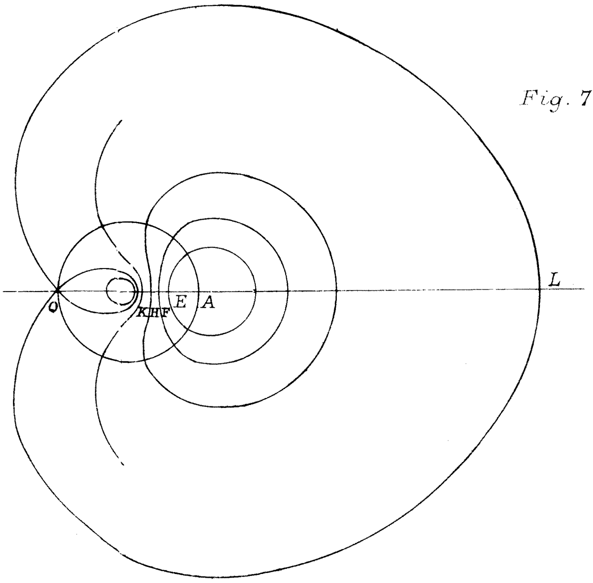
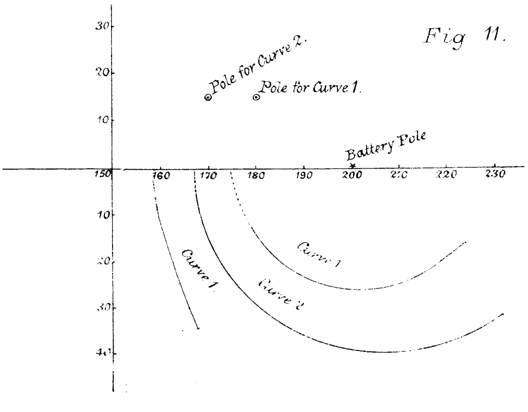


Fig 11.



W. West & C° imp.

W. G. Adams.

Fig. 2.

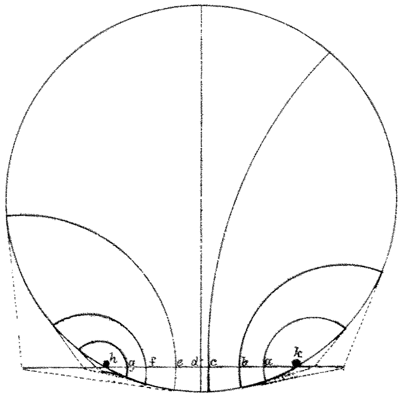


Fig. 3

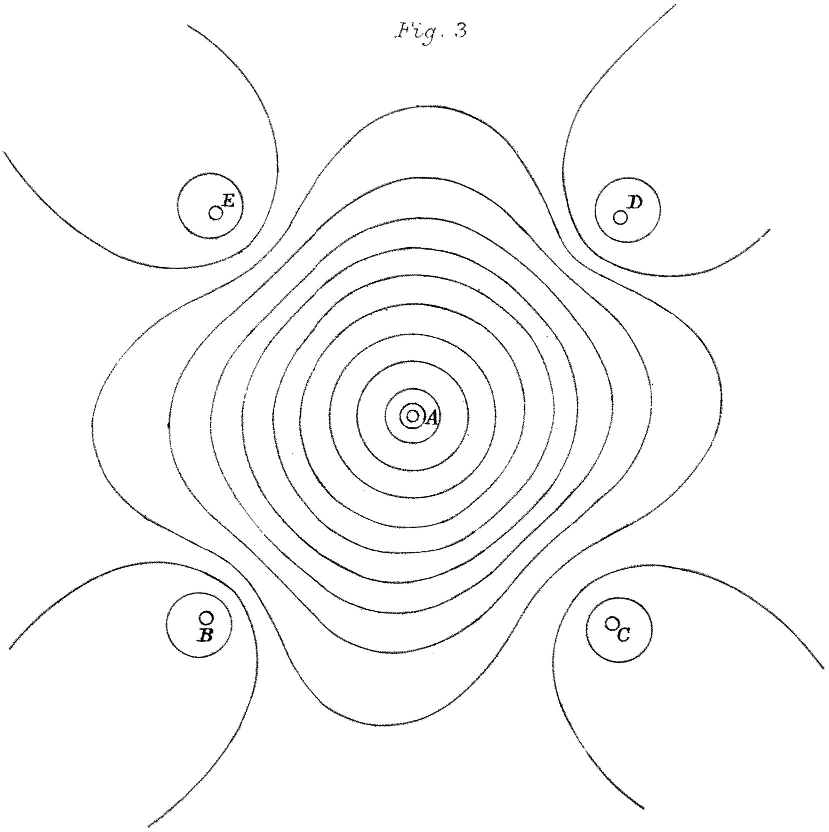
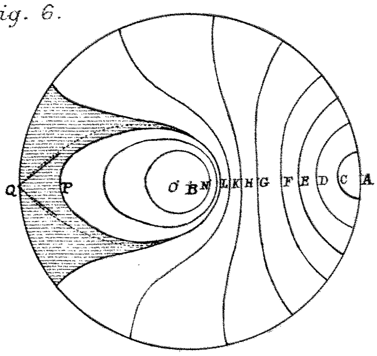


Fig. 6.



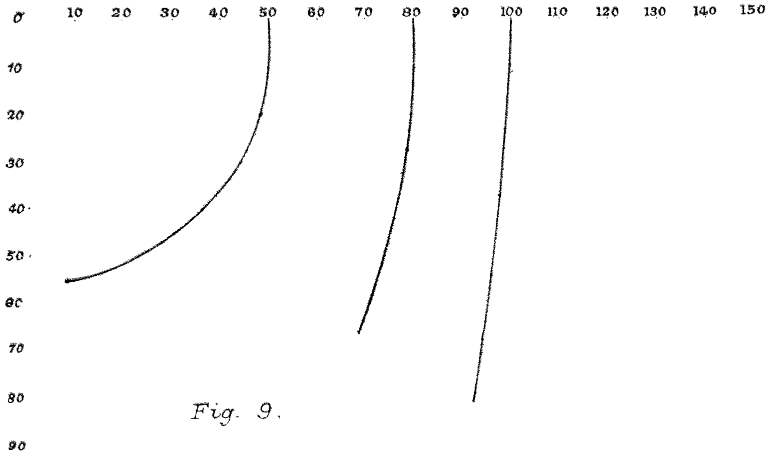


Fig. 9.

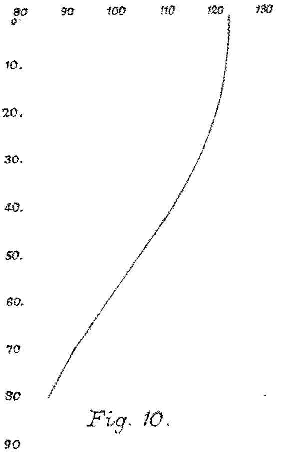
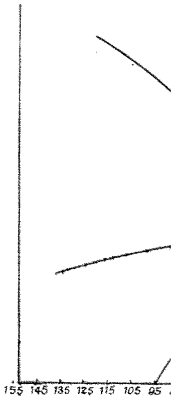
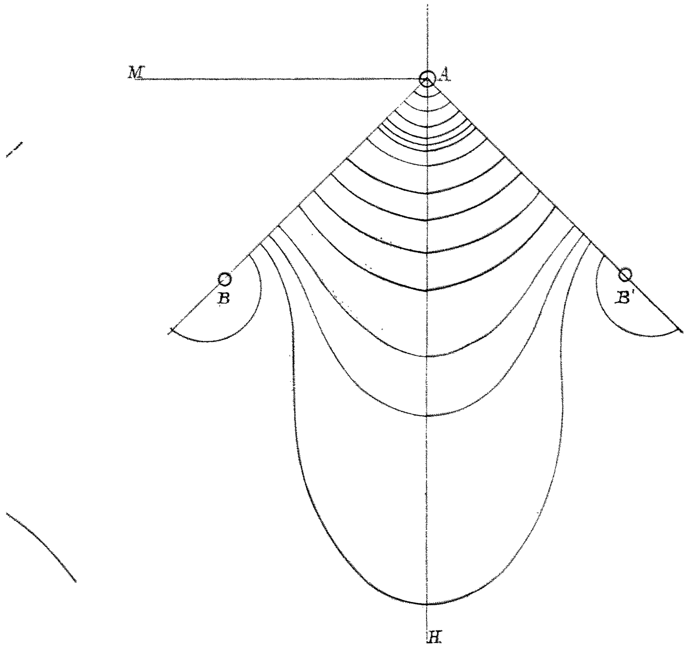


Fig. 10.



W.H.Wesley lith.

Fig. 4.



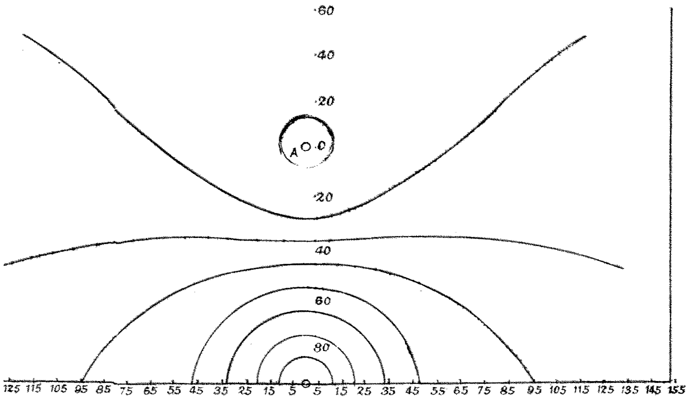


Fig. 12.

W. West & C° imp.

Fig. 1.

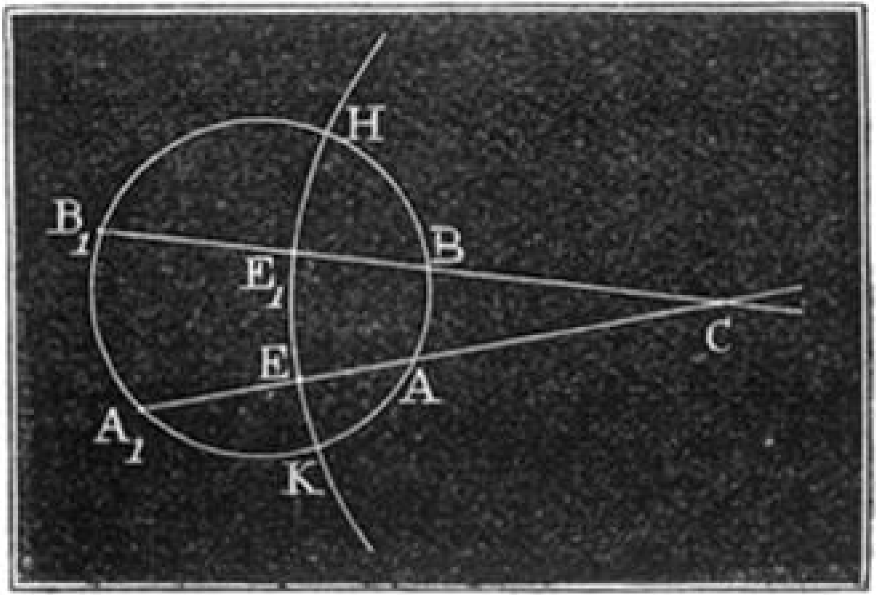


Fig. 2.

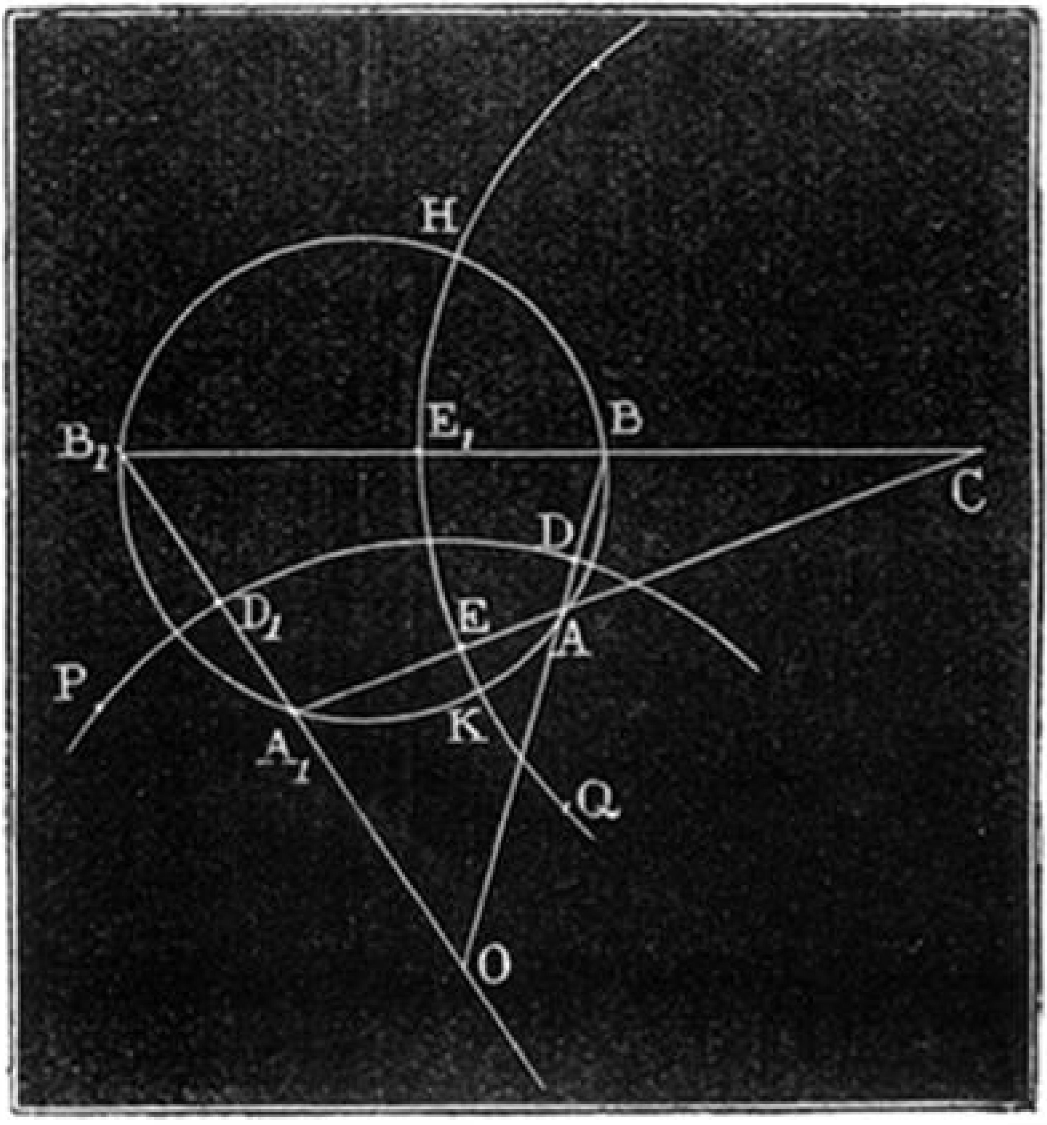


Fig. 3.

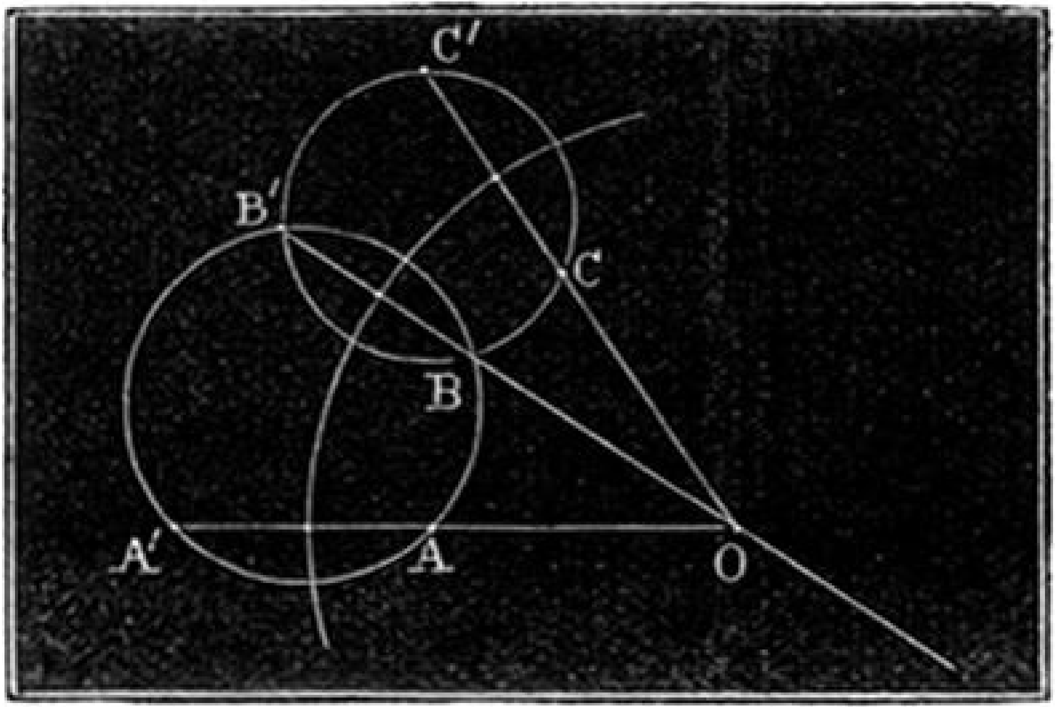


Fig. 4.

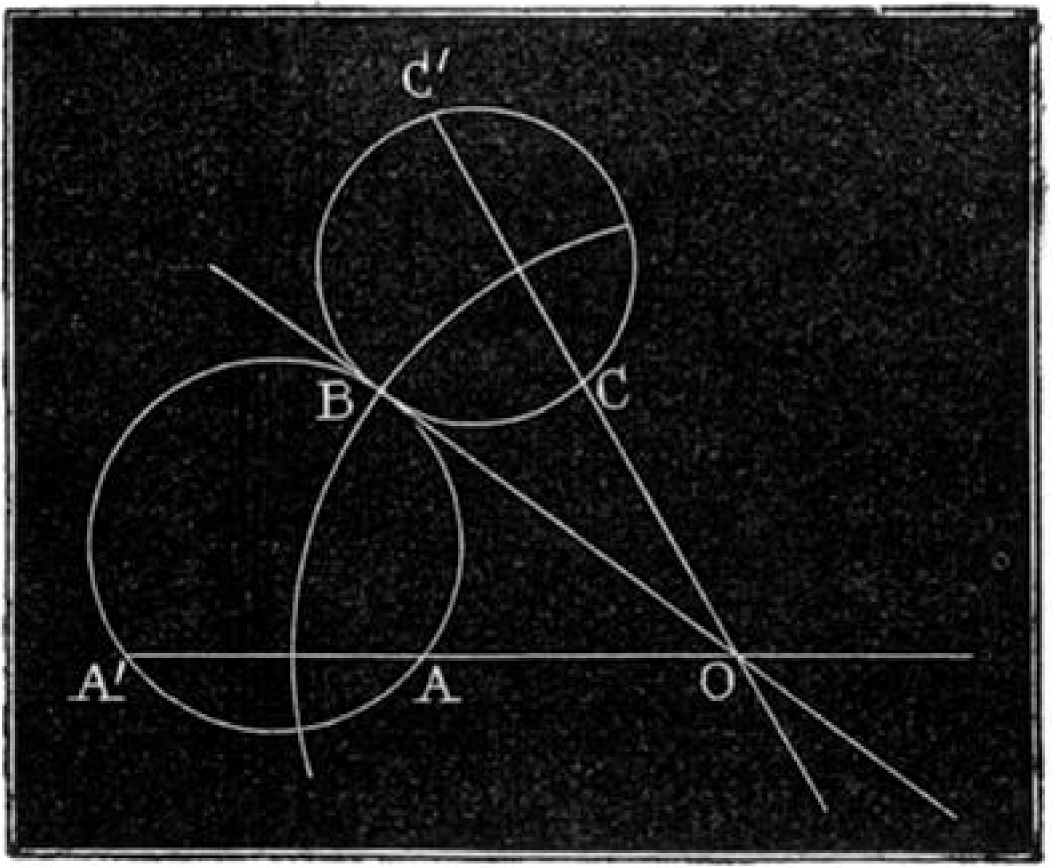
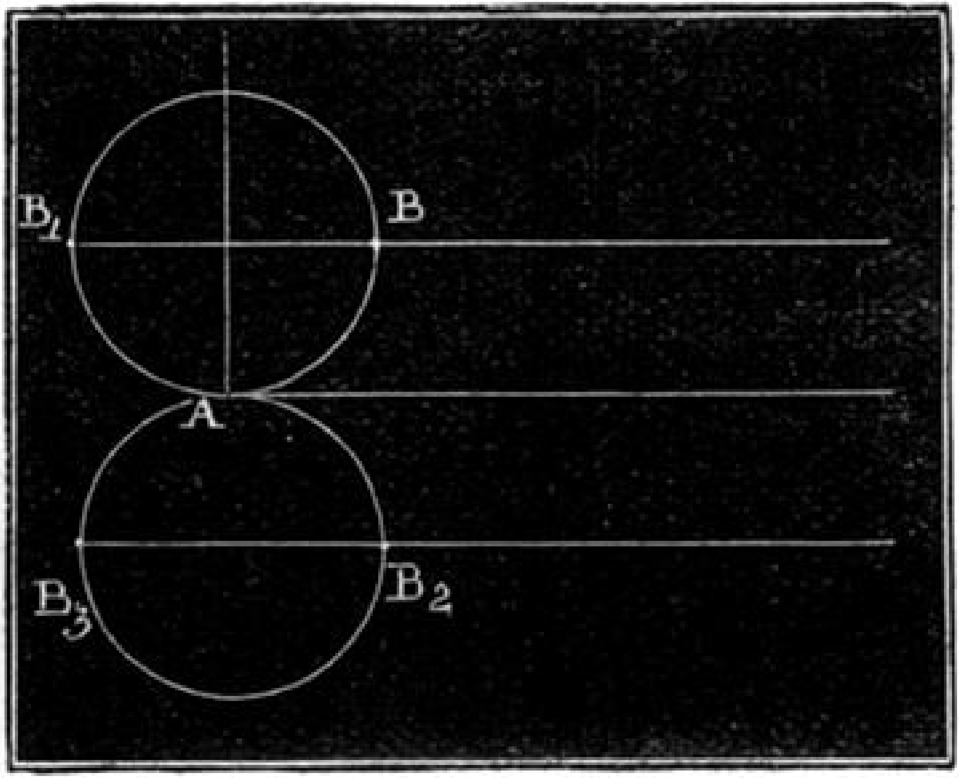


Fig. 5.



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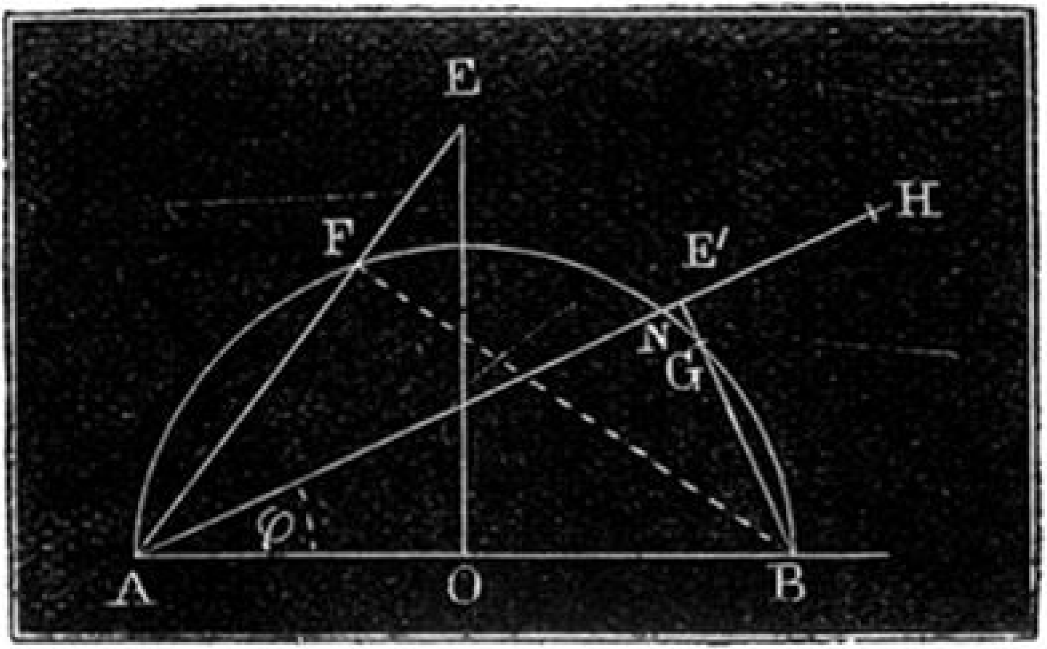
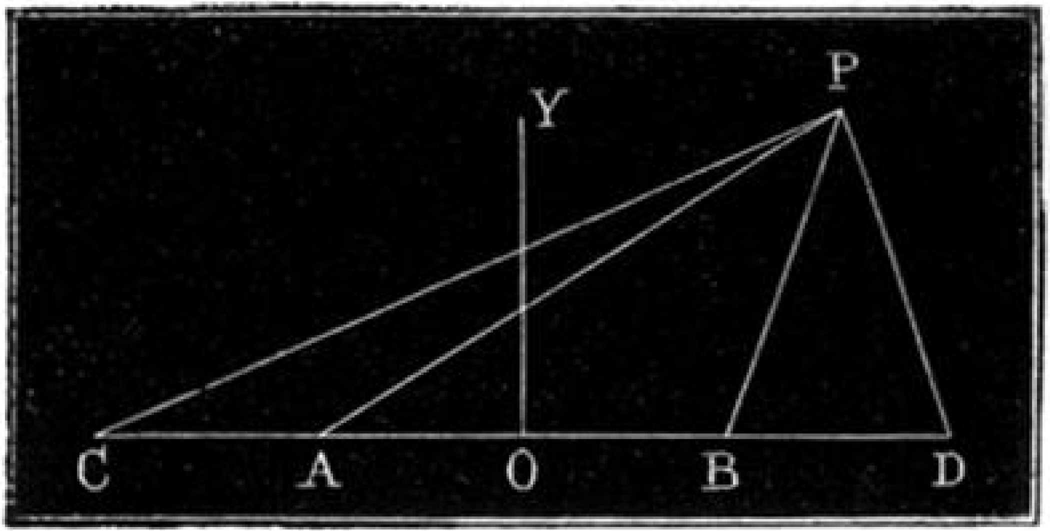


Fig. 7.



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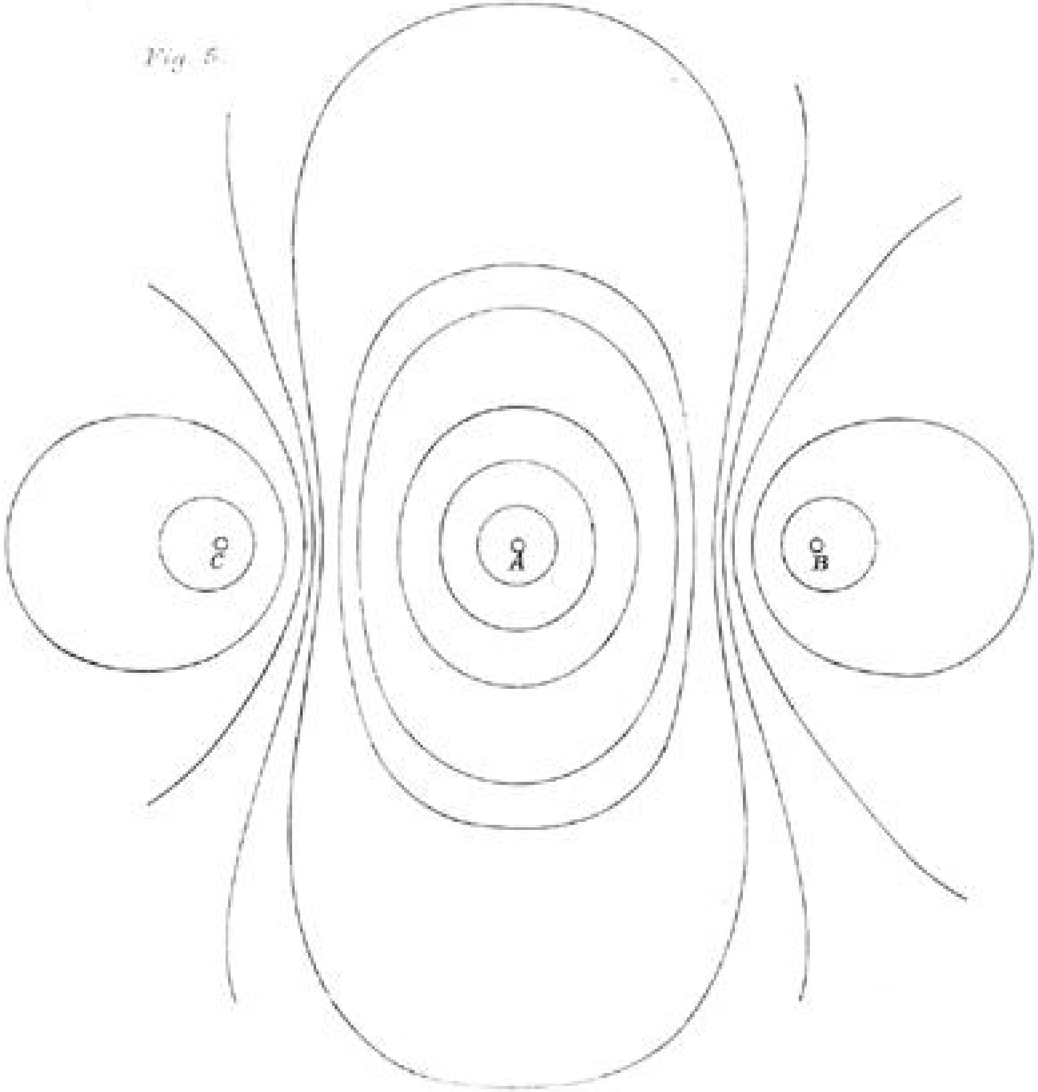


Fig. 1

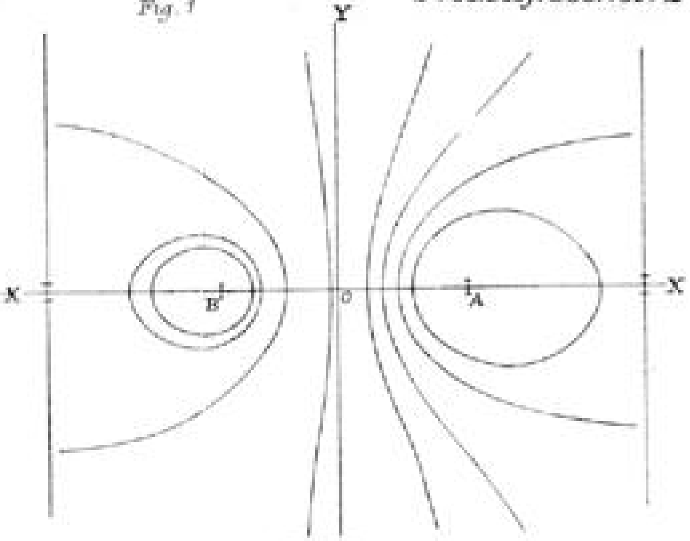


Fig. 7.

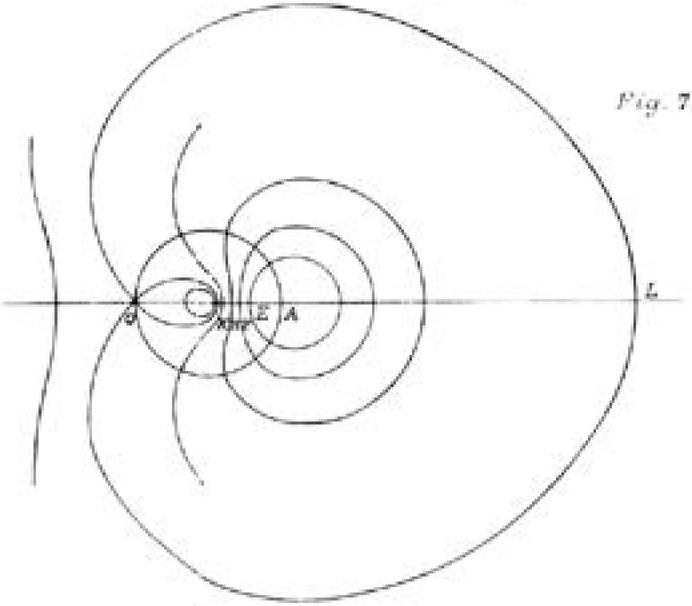


Fig. 8.

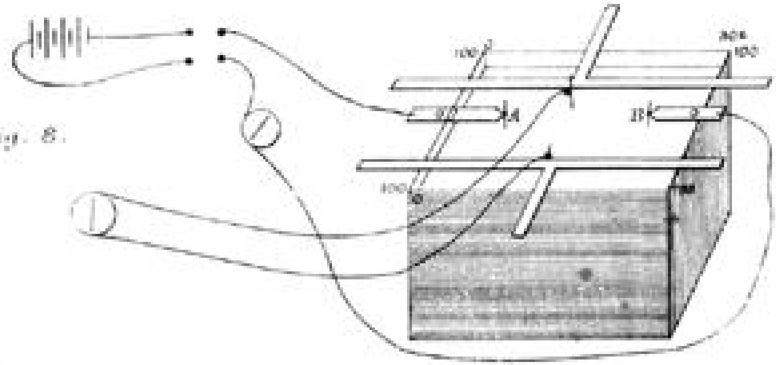
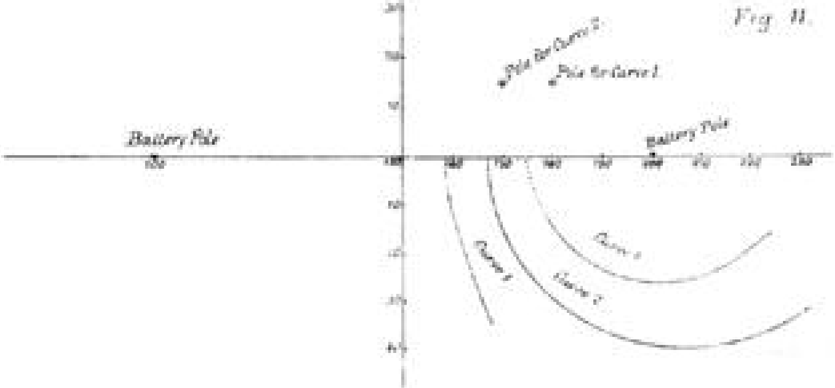


Fig. 11.



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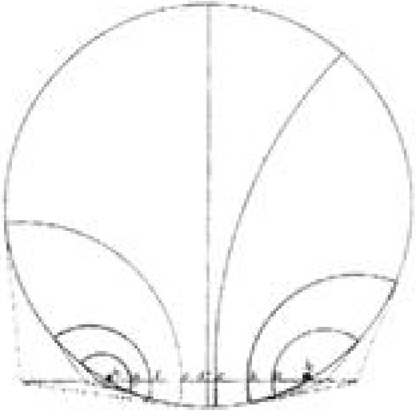

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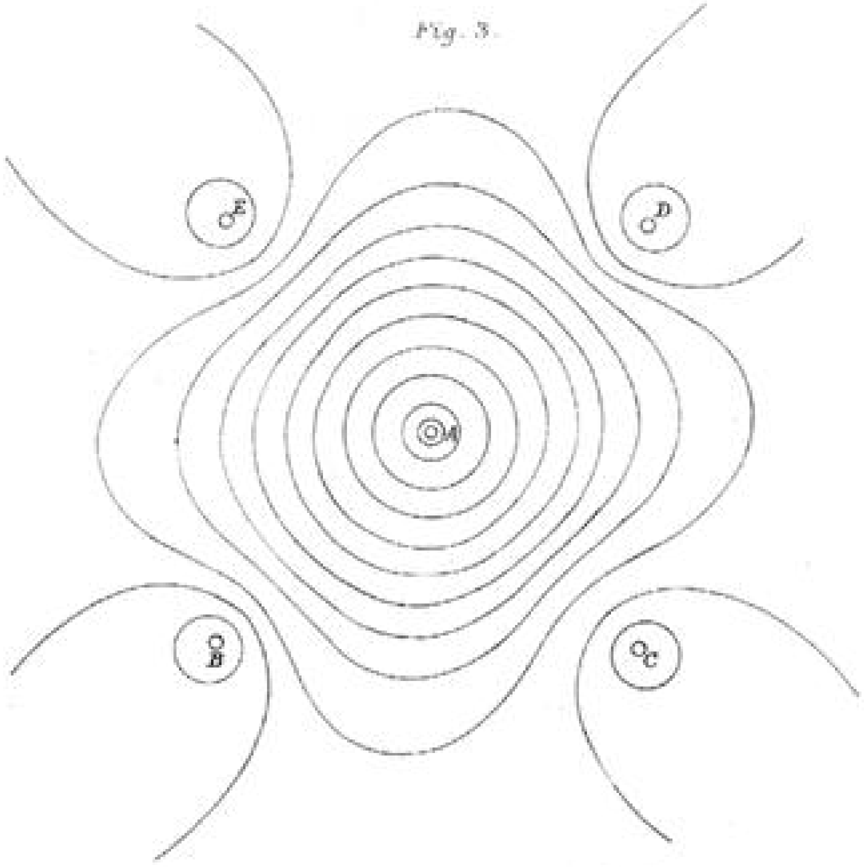

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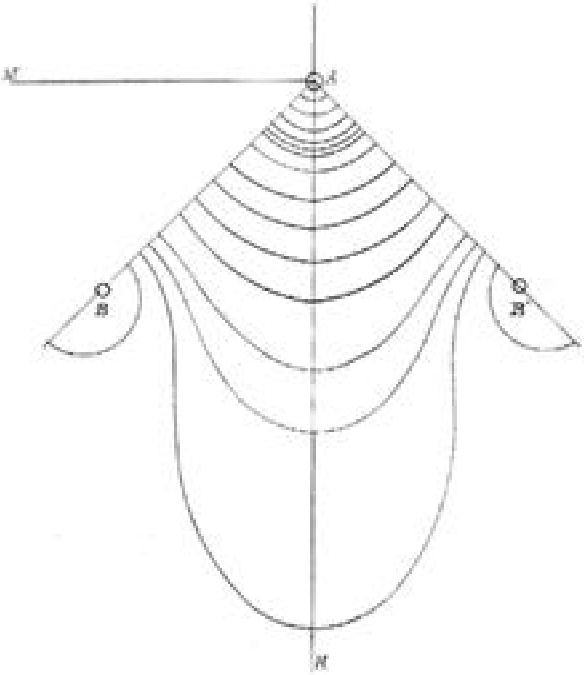
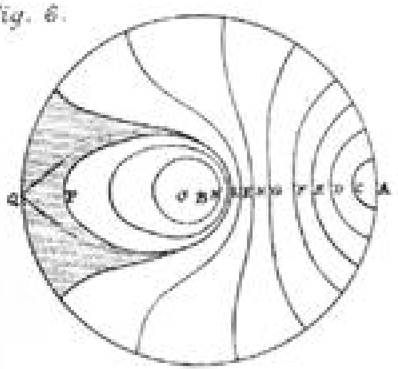


Fig. 6.



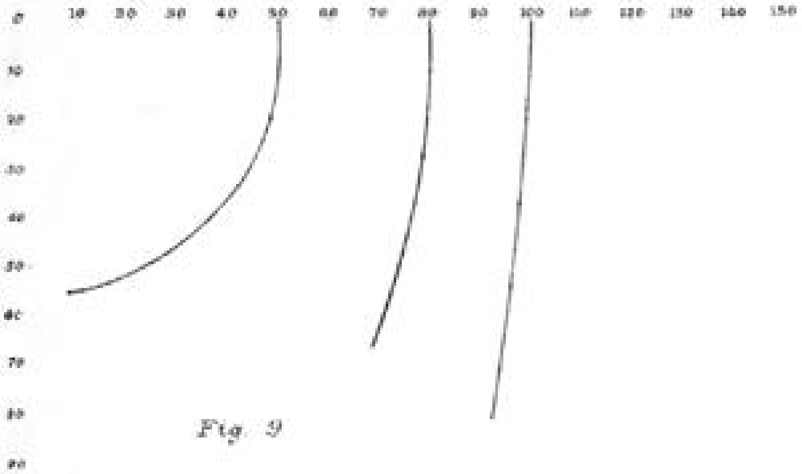


Fig. 9

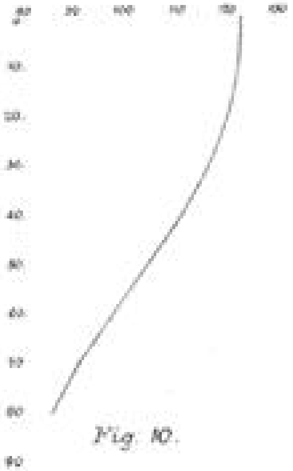


Fig. 10.

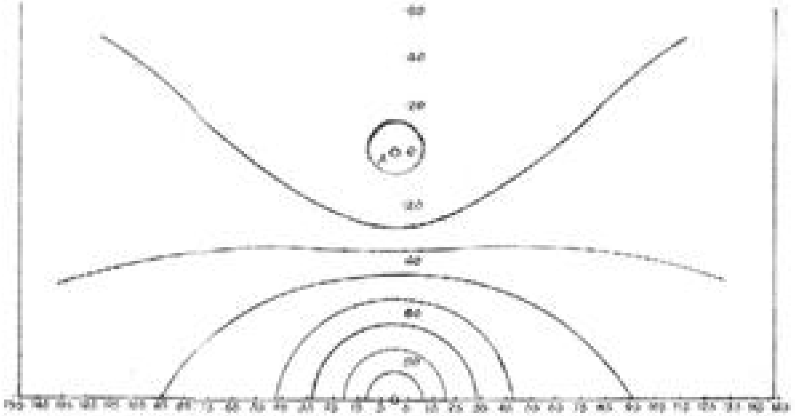


Fig. 12.